Splittings and finite basis theorems
Part II: Complete lattices of subquasivarieties

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This is the second part of a two-part talk, and we use some definitions and notations from the Part I.

For variety V or quasivariety Q, $\Lambda_v(\mathsf{V})$ and $\Lambda_q(\mathsf{Q})$ denote the complete lattices of all subvarieties and all subquasivarieties, of V or Q; as every variety is a quasivariety the notation $\Lambda_q(\mathsf{V})$ also makes sense. The following observation shows the relations between splittings in $\Lambda_v(\mathsf{V})$ and $\Lambda_q(\mathsf{V})$.

Theorem 1. *Let* W *be a variety,* $\mathsf{V} \in \Lambda_v(\mathsf{W})$ *and* $\mathsf{Q} = \mathbf{Q}(\mathbf{F}_{\mathsf{V}}(\omega))$*. Then* V *splits* $\Lambda_v(\mathsf{W})$ *if and only if* Q *splits* $\Lambda_q(\mathsf{W})$*.*

If Q is a quasivariety, algebra $\mathbf{A} \in \mathsf{Q}$ is **Q-irreducible** if there are two elements $a, b \in \mathbf{A}$ such that for any distinct from identity congruence θ of $\mathbf{A}$, if $\mathbf{A}/\theta \in \mathsf{Q}$, then $(a, b) \in \theta$. And $\mathbf{A}$ is finitely Q-presentable if there is is a compact congruence θ of $\mathbf{F}_{\mathsf{Q}}(n)$ such that $\mathbf{A} \cong \mathbf{F}_{\mathsf{Q}}(n)/\theta$.

Similarly to splitting varieties (cf., e.g., [1]) the following holds for splitting quasivarieties.

Theorem 2. *Suppose that* K *is a quasivariety and the pair* $(\mathsf{Q}, \mathsf{Q}^*)$ *splits* $\Lambda_q(\mathsf{K})$*. Then*

1) Q^* *is axiomatized relative to* K *by any quasiequation* φ *such that* $\mathsf{Q}^* \models \varphi$ *and* $\mathsf{Q} \not\models \varphi$*;*
2) Q *is generated by a single finitely generated* Q*-irreducible algebra* $\mathbf{A}$*;*
3) Q *is generated by a single finitely* Q*-presented algebra* $\mathbf{A}$*.*

Among quasiequations mentioned in (1) there always is a **Q-irreducible quasiequation** φ: if $\Phi \models_{\mathsf{Q}} \varphi$, then there is $\varphi' \in \Phi$ such that $\varphi' \models_{\mathsf{Q}} \varphi$; the Q-irreducible quasiequation defining relative to Q the co-splitting subquasivariety is called a **splitting quasiequation**.

The biggest difference between splittings in the lattices of varieties and quasivarieties is that if a pair $(\mathsf{V}, \mathsf{V}^*)$ splits $\Lambda_v(\mathsf{W})$, the V-irreducible algebra generating V is subdirectly irreducible and thus it is W-irreducible. For quasivarieties it is not the case: if pair $(\mathsf{Q}, \mathsf{Q}^*)$ splits $\Lambda_q(\mathsf{K})$, Q may not be generated by any K-irreducible algebras. This observation justifies the following definitions: algebra $\mathbf{A}$ is **self-irreducible** if it is $\mathbf{Q}(\mathbf{A})$-irreducible; algebra $\mathbf{A}$ is a **splitting algebra** in Λ if it is finitely generated self-irreducible and quasivariety $\mathbf{Q}(\mathbf{A})$ splits Λ; and $\mathbf{A}$ is a **strong splitting algebra** if it is a splitting algebra and in addition it is K-irreducible, where K is the top element of Λ. For a quasivariety K by K_{spl} we denote the class of all algebras splitting $\Lambda_q(\mathsf{K})$. On K_{spl} we also define a quasi-order by letting for any $\mathbf{A}, \mathbf{B} \in \mathsf{K}_{spl}$, $\mathbf{A} \leq \mathbf{B} \leftrightharpoons \mathbf{Q}(\mathbf{A}) \subseteq \mathbf{Q}(\mathbf{B})$; and this quasi-order can be easily converted into a partial order on the cosets.

The notion of separability was defined in the Part I. For instance, if quasivariety Q and all its subquasivarieties have the finite embeddability property (FEP for short), that is if each quasivariety from $\Lambda_q(\mathsf{Q})$ is generated by its finite members, then $\Lambda_q(\mathsf{Q})$ is separable.

Theorem 3. *Let* Λ *be a complete lattice of quasivarieties and* K *be its top element. If* $\mathsf{Q} \in \Lambda$ *is separable, then it has a basis consisting of splitting quasiequations relative to* K*. Thus, if* Λ *is separable, then every member of* Λ *has a basis relative to* K *consisting of splitting quasiequations.*

A quasivariety Q is **primitive** if every its subquasivariety can be defined relative to Q by a set of identities, i.e. for every $\mathsf{Q}' \in \Lambda_q(\mathsf{Q})$, $\mathsf{Q}' = \mathsf{Q} \cap \mathbf{V}(\mathsf{Q}')$, where $\mathbf{V}(\mathsf{Q}')$ is the variety generated by Q'. The primitive quasivarieties are the algebraic counterparts of hereditarily structurally complete finitary structural consequence relations. And Q is **weakly primitive** if in every $\mathsf{Q}' \in \Lambda_q(\mathsf{Q})$, every algebra $\mathbf{A} \in \mathsf{Q}'$ is a subdirect product of Q-irreducible algebras from Q'.

Theorem 4. *Every primitive quasivariety is weakly primitive. Moreover, quasivariety Q is weakly primitive if and only if every self-irreducible algebra in Q is Q-irreducible.*

A quasivariety Q is **weakly tame** if every finitely generated Q-irreducible algebra in Q is Q-splitting (and thus it is strong Q-splitting). For instance, every quasivariety of finite type with the FEP (hence any locally finite quasivariety of finite type) is weakly tame.

Corollary 5. *If Q is weakly tame and weakly primitive, then $\mathsf{Q} = \mathbf{Q}(\mathsf{Q}_{spl})$.*

If $\mathsf{Q}' \subseteq \mathsf{Q}$, we define $I[\mathsf{Q}', \mathsf{Q}] = \{\mathsf{Q}'' : \mathsf{Q}' \subseteq \mathsf{Q}'' \subseteq \mathsf{Q}\}$.

Theorem 6. *Let Q be weakly primitive, weakly tame quasivariety of finite type and $\mathsf{Q}' \subseteq \mathsf{Q}$ such that every quasivariety in $I[\mathsf{Q}', \mathsf{Q}]$ has the FEP. Then the following are equivalent:*

1) *every $\mathsf{Q}'' \in I[\mathsf{Q}', \mathsf{Q}]$ has a finite basis relative to Q;*
2) *$I[\mathsf{Q}', \mathsf{Q}]$ is countable;*
3) *$\mathsf{Q}_{spl} \setminus \mathsf{Q}'_{spl}$ has no infinite antichain;*
4) *$I[\mathsf{Q}', \mathsf{Q}]$ enjoys the descending chain condition.*

Corollary 7. *If Q is weakly primitive, of finite type and finitely generated then $\Lambda_q(\mathsf{Q})$ is finite and all its subquasivarieties have a finite basis relative to Q.*

Proof. The proof follows from the observation that Q has just a finite (up to isomorphism) set of strong Q-splitting algebras. □

Corollary 8. *If Q is primitive, finitely axiomatizable and of finite type, then every finitely generated subquasivariety of Q is finitely axiomatizable.*

Remark 9. Corollary 7 can be seen as a version of Baker's Finite Basis Theorem for quasivarieties; our version differs from the one in [2], in that we drop relative congruence distributivity and add weak primitivity.

Note that primitivity is essential in Corollary 7. In [3] (also see [4, Section 4.5]) Rybakov gave an example of finite Heyting algebra $\mathbf{A}$ with $\mathbf{Q}(\mathbf{A})$ not having a finite basis relative to variety of all Heyting algebras and therefore, relative to $\mathbf{V}(\mathbf{A})$. We note that $\Lambda_q(\mathbf{V}(\mathbf{A}))$ is infinite, while $\Lambda_v(\mathbf{V}(\mathbf{A}))$ is finite. Rybakov's example also shows that congruence distributive varieties may have subquasivarieties which are not relatively congruence distributive.

References

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