# Splittings and finite basis theorems 1: splittings of a lattice

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This is the first part of a two-part talk. We study the splittings in the lattices of quasivarieties, which, as it is very well known, are often the algebraic semantics for finitary structural consequence relations. The splittings in lattices of varieties were extensively studied, and this quest was instigated by seminal paper [2]. In addition, we do not restrict ourselves to complete lattices of all subquasivarieties of a given quasivariety; instead, we often consider an arbitrary complete lattice of quasivarieties of a given type. This, for instance, allows us to study intervals $\{\mathsf{Q}' : \mathbf{Q}(\mathbf{F}_{\mathsf{V}}(\omega)) \subseteq \mathsf{Q}' \subseteq \mathsf{V}\}$, where $\mathsf{V}$ is a variety, which sheds light on the properties of the sets of rules admissible in the logic having $\mathsf{V}$ as its algebraic semantics. Before dealing with applications it is convenient to lay down some general theory about splittings of a lattice. Let $\mathbf{L}$ be any lattice; an ordered pair $(a, b)$ of elements of $\mathbf{L}$ such that $a \not\leq b$ is a **splitting pair** (**splitting** for short) if for every $c \in \mathbf{L}$, either $a \leq c$, or $c \leq b$. We call $a$ a **splitting element** and $b$ a **co-splitting element**; if $(a, b)$ is a splitting of $\mathbf{L}$ we also will say that the pair $(a, b)$ **splits L**. The concept of splitting pair originated in [3]; there Whitman defined a splitting of a lattice to be a pair $(F, I)$ where $F$ and $I$ are a filter and an ideal of $\mathbf{L}$ respectively, and $L$ is the disjoint union of $F$ and $I$. Therefore the concept we have introduced is akin to a *principal splitting* in [3]. Given a lattice $\mathbf{L}$ we say that $a \in L$ is **completely join prime** if for all $X \subseteq L$, if $\bigvee X$ exists and $a \leq \bigvee X$, then there is an $x \in X$ with $a \leq x$. A **completely meet prime** element of $\mathbf{L}$ is defined dually. The following facts are either straifghforward or have been shown in [3]:

1. If $(a_1, b_1), (a_2, b_2)$ split $\mathbf{L}$, then $a_1 \leq a_2$ if and only if $b_1 \geq b_2$ and $a_1 < a_2$, if and only if and only if $b_1 > b_2$.
2. If $(a, b), (a, c)$ split $\mathbf{L}$, then $b = c$ and if $(a, c), (b, c)$ split $\mathbf{L}$, then $a = b$. Therefore if $a$ is a splitting element in $\mathbf{L}$ there is a unique co-splitting element $a^*$ (called the **conjugate** of $a$) such that $(a, a^*)$ splits $\mathbf{L}$; similarly for any co-splitting element $b$ there is a unique splitting element $a \in L$ with $b = a^*$.
3. If $a$ is splitting in $\mathbf{L}$, then $a$ is completely join prime and if $\mathbf{L}$ is complete, then the converse holds.
4. If $a$ is co-splitting in $\mathbf{L}$, then $a$ is completely meet prime and if $\mathbf{L}$ is complete, then the converse holds.
5. Let $\mathbf{M}$ be a complete sublattice of $\mathbf{L}$; if $a \in M$ is splitting in $\mathbf{L}$, then $a$ is splitting in $\mathbf{M}$ with conjugate element $a^* = \bigvee\{b \in M : a \not\leq b\}$.

Antichains of splitting elements are important in a lattice.

**Theorem 1.** *[1] Let $\mathbf{L}$ be a lattice and $S$ be an infinite antichain of splitting elements. Then,*

1. *$\mathbf{L}$ contains continuum many sublattices;*
2. *$\mathbf{L}$ has infinite ascending and descending chains of elements;*
3. *if $\mathbf{L}$ is complete, then $\mathbf{L}$ is not countable.*

Let $\mathbf{L}$ be a lattice. Element $a \in L$ is **decomposable** if there is a subset $\mathsf{S} \subseteq L$ of completely meet prime elements such that $a = \bigwedge S$; an element $a \in L$ is **join decomposable** if there is a subset $\mathsf{S} \subseteq L$ of completely join prime elements such that $a = \bigvee S$. A decomposition $a = \bigwedge S$ is **irredundant** if for every $b \in S$, $a \neq \bigwedge(S \setminus \{b\})$; a join decomposition is **irredundant** if the dual property holds. Every completely meet prime (join prime) element has a **trivial irredundant decomposition (join decomposition)** consisting of itself. If the lattice $\mathbf{L}$ is complete, then a (join) decomposition is a decomposition into splitting (co-splitting) elements of $\mathbf{L}$.

**Proposition 2.** *Let* $\mathbf{L}$ *be a lattice and* $S \subseteq \mathbf{L}$ *be a set of meet-prime elements. Then the decomposition* $a = \bigwedge S$ *is irredundant if and only if* $S$ *is an antichain.*

An element $b$ of lattice $\mathbf{L}$ is **separable** if for every $c \in \mathbf{L}$, if $c \not\leq b$, then there is a splitting element $a$ such that $b \leq a^*$ and $a \leq c$. It follows that the top element of $\mathbf{L}$, if any, is always separable. Dually we say that $c \in L$ is **co-separable** if for every $b \in \mathbf{L}$, if $c \not\leq b$, then there is a splitting element $a$ such that $b \leq a^*$ and $a \leq c$. It follows that the bottom element of $\mathbf{L}$, if any, is always co-separable. A lattice is **separable** if all its elements are separable.

**Theorem 3.** *[1] For a complete lattice* $\mathbf{L}$ *the following are equivalent:*

1. $\mathbf{L}$ *is separable;*
2. *every element different from the top is decomposable into a meet of co-splitting elements;*
3. *every element different from the bottom is join decomposable into a join of splitting elements.*

**Theorem 4.** *[1] Let* $\mathbf{L}$ *be a complete separable lattice and* $\mathsf{S}_\mathbf{L}$ *be the set of all its splitting elements. If* $\mathsf{S}_\mathbf{L}$ *is countable and enjoys the descending chain condition, then the following are equivalent:*

1. $\mathbf{L}$ *is at most countable;*
2. $\mathsf{S}_\mathbf{L}$ *has no infinite antichains;*
3. *each element of* $\mathbf{L}$ *has a finite irredundant decomposition.*

It is well-known that the class of all quasivarieties and the class of all varieties of algebras of a given type form complete lattices; we are interested in complete sublattices of those lattices. If $\mathsf{Q}$ is any quasivariety, then all the subquasivarieties of $\mathsf{Q}$ form a complete lattice $\Lambda_q(\mathsf{Q})$. If $\mathsf{V}$ is variety then all the subvarieties of $\mathsf{V}$ form a complete lattice $\Lambda_v(\mathsf{V})$; since $\mathsf{V}$ is also a quasivariety the notation $\Lambda_q(\mathsf{V})$ makes sense. Observe however that $\Lambda_v(\mathsf{Q})$ and $\Lambda_q(\mathsf{V})$ may be quite different; there are examples of varieties whose lattice of subvarieties and whose lattice of subquasivarieties are infinite; there are also examples in which the lattice of subvarieties is countable (even finite!) but the lattice of subquasivarieties is uncountable. Moreover the notation $\Lambda_v(\mathsf{Q})$ for a quasivariety $\mathsf{Q}$ also makes sense; however in this case the lattice may not be complete, in that there are examples of quasivarieties $\mathsf{Q}$ in which there is no largest variety contained in $\mathsf{Q}$. Now we have set up the playground for applications, that will be dealt with in Part 2.

# References

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