Three Theorems on Idempotent Semifields

George Metcalfe and Simon Santschi

Mathematical Institute, University of Bern, Switzerland
{george.metcalfe,simon.santschi}@unibe.ch

We call an algebraic structure $\langle S, \vee, \cdot, \mathrm{e}\rangle$ an *idempotent semiring* if

(i) $\langle S, \cdot, \mathrm{e}\rangle$ is a monoid;

(ii) $\langle S, \vee\rangle$ is a semilattice (i.e., an idempotent commutative semigroup); and

(iii) $a(b \vee c)d = abd \vee acd$ for all $a, b, c, d \in S$,

and an *idempotent semifield* if, additionally, $\langle S, \cdot, \mathrm{e}\rangle$ is the monoid reduct of a group. These structures play an important role in many areas of mathematics, including idempotent analysis, tropical geometry, formal language theory, and mathematical logic (see [6] for details). Other definitions of an idempotent semiring (also known as a *dioid* or an *ai-semiring*) may be found in the literature — in particular, an idempotent semifield may be defined without e in the signature, or with an extra constant symbol 0 interpreted as the neutral element of $\vee$, where $\langle S\backslash\{0\}, \cdot, \mathrm{e}\rangle$ is a group — but our results extend also to these settings.

Expanding an idempotent semifield $\langle S, \vee, \cdot, \mathrm{e}\rangle$ with the group inverse operation $^{-1}$ and lattice meet operation $\wedge$ defined by $a \wedge b := (a^{-1} \vee b^{-1})^{-1}$ produces a *lattice-ordered group* (or *ℓ-group*). Moreover, idempotent semifields are precisely the semiring reducts of ℓ-groups. In this work, which is developed in full in [8], we answer three open problems about equational theories of classes of idempotent semifields. These problems have been solved for classes of ℓ-groups, but restricting to fewer operations requires new proof methods and yields notably different results.

Let K be any class of $\mathcal{L}$-algebras for some signature $\mathcal{L}$, and call it *non-trivial* if at least one of its members is non-trivial, i.e., has more than one element. The *equational theory* $\mathrm{Eq}(\mathsf{K})$ of K is the set of all $\mathcal{L}$-equations $s \approx t$ such that $\mathsf{K} \models s \approx t$. A *basis* for this equational theory is a set of equations $\Sigma \subseteq \mathrm{Eq}(\mathsf{K})$ such that every equation in $\mathrm{Eq}(\mathsf{K})$ is a logical consequence of Σ. If $\mathrm{Eq}(\mathsf{K})$ has a finite basis, then K is said to be *finitely based*. Our first theorem is a complete answer to the finite basis problem for idempotent semifields. Although countably infinitely many equational theories of ℓ-groups have a finite basis (see, e.g., [2]), we prove, extending previous results obtained in [1], that:

Theorem A. *There is no non-trivial class of idempotent semifields that is finitely based.*

Our second theorem concerns the number of equational theories of classes of idempotent semifields. Using a technique of 'inverse elimination' introduced in [3, Section 4] to translate between equations in the different signatures, we obtain a one-to-one correspondence between a family of equational theories of ℓ-groups that is known to be uncountable (see [7]) and equational theories of certain classes of idempotent semifields, thereby proving:

Theorem B. *There are continuum-many equational theories of classes of idempotent semifields.*

The final theorem concerns the complexity of deciding equations in the class of idempotent semifields. The equational theory of the class of ℓ-groups is known to be co-NP-complete [5, Theorem 8.3] and we prove that this is also the case for the restricted signature, that is:

Theorem C. *The equational theory of the class of idempotent semifields is co-NP-complete.*

Using this result together with [4, Theorem 2], which relates the validity of equations in ℓ-groups to the existence of right orders on free groups, we also obtain the following:

Corollary. *Let* $\mathbf{F}(X)$ *be the free group over a set* X *with* $|X| \geq 2$*. Then the problem of checking for* $s_1, \dots, s_n \in \mathrm{F}(X)$ *if there exists a right order* $\leq$ *on* $\mathbf{F}(X)$ *satisfying* $\mathrm{e} < s_1, \dots, \mathrm{e} < s_n$ *is NP-complete.*

References

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