Topological Duality for Distributive Lattices: Theory and Applications

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This is a talk about the book [1] on topological duality theory for bounded distributive lattices recently published by Cambridge University Press, and it will be presented jointly by the authors. The purpose of the talk will be to give an overview of the content and potential uses of the book in teaching and research, and to sound out the audience on potentially useful additional resources we could put on the web. In the rest of this abstract, we draw from the book's preface to give a quick overview of its contents as we plan to present them in the conference presentation.

The book is a course on Stone-Priestley duality theory, with applications to logic and the foundations of computer science. Our target audience includes both graduate students and researchers in mathematics and computer science. The main aim of the book is to equip the reader with the theoretical background necessary for reading and understanding current research in duality and its applications. We have aimed to be didactic rather than exhaustive, while we did give technical details whenever they are necessary for understanding what the field is about.

A unique feature of the book is that, in addition to developing general duality theory for distributive lattices, we also show how it applies in a number of areas within the foundations of computer science, namely, modal and intuitionistic logics, domain theory and automata theory. The use of duality theory in these areas brings to the forefront how much their underlying mathematical theories have in common. It also prompts us to upgrade our treatment of duality theory with various enhancements that are now commonly used in state-of-the-art research in the field. Most of these enhancements make use of *operators* on a distributive lattice: maps between lattices that only preserve part of the lattice structure. We give a textbook exposition of the theory of lattices with operators, and dualities for them, as it was developed in the second half of the 20th century. Our exposition of the theory also treats several of its by now classical applications, such as those to free distributive lattices, quotients and subspaces, implication-type operators, Heyting algebras and Boolean envelopes.

In the first chapters of the book, we keep the use of category theory to a minimum. We then set the results in the more abstract and general framework of category theory. This development also allows us to show how Priestley's duality fits well in a more general framework for the interaction of topology and order, which had been developed by Nachbin shortly before. We show how the various classes of topological spaces with and without order, introduced by Stone, Priestley and others, all relate to each other, and how they are in duality with distributive lattices and their infinitary variant, frames.

The book ends with an extended exposition of two more modern applications of duality theory to theoretical computer science, namely to domain theory and to automata theory. The domain theory that we develop is organized around three separate results: Hoffmann-Lawson duality; the characterization of those dcpos and domains, respectively, that fall under Stone duality; and Abramsky's celebrated 1991 Domain Theory in Logical Form paper. The duality-theoretic approach to automata theory that we develop in the book originates in work due to the first author with Grigorieff and Pin. It is organized around a number of related results, namely:

finite syntactic monoids can be seen as dual spaces, and the ensuing effectivity of this powerful invariant for regular languages; the free profinite monoid is the dual of the Boolean algebra of regular languages expanded with residuation operations and, more generally, topological algebras on Boolean spaces are duals of certain Boolean algebras extended by residual operations. As an extended application example, we use duality to give a profinite equational characterization for the class of piecewise testable languages; and we end by discussing a characterization of those profinite monoids for which the multiplication is open.

These two applications, and in particular the fact that we treat them in one place, as applications of a common theory, are perhaps the most innovative and special aspects of this book. Domain theory is the most celebrated application of duality in theoretical computer science and our treatment is entirely new. Automata theory is a relatively new application area for duality theory and has never been presented in textbook format before. More importantly, both topics are currently at the forefront of active research seeking to unify semantic methods with more algorithmic topics in finite model theory. While previous treatments remained focused on the point of view of domains/profinite algebra, with duality theory staying peripheral, a shared innovative aspect of the presentations of these topics in this book is that both are presented squarely as applications of duality.

Finally, a completely original contribution of this book, which emerged during its writing, precisely thanks to our treatment of the two topics as an application of a common theory, is the fact that a notion we call "*preserving joins at primes*" turns out to be central in both the chapter on domain theory and in that on automata theory. This notion was introduced in the context of automata theory and topological algebra by the first author in 2016; its application to domain theory is new to this book and reflects a key insight of Abramsky's Domain Theory in Logical Form. We believe this point to be an exciting new direction for future research in the field that we hope some readers of the book will be inspired to take up.

References

[1] Gehrke M., van Gool S. *Topological Duality for Distributive Lattices: Theory and Applications*. Cambridge University Press; 2024. https://www.cambridge.org/9781009349697