Unification for temporal logic via duality and automata

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The *unifiability problem* for an equational theory $\mathcal{E}$ is the decision problem that asks, given as input two $\mathcal{E}$-terms

$$A(x_1, \ldots, x_n) \quad \text{and} \quad B(x_1, \ldots, x_n) \ , \tag{1}$$

to decide whether or not there exists a syntactic substitution of the variables by terms, $x_i \mapsto C_i$, such that the resulting terms $A(C_1, \ldots, C_n)$ and $B(C_1, \ldots, C_n)$ are equal modulo the theory $\mathcal{E}$.

We establish decidability of the unifiability problem for a logic we call $\mathbf{X}$, the next-fragment of linear temporal logic, enriched with an arbitrary fixed number of propositional constants $\mathbf{c}_1, \ldots, \mathbf{c}_n$. In algebraic terms, the equational theory $\mathcal{E}_{\mathbf{X}}$ associated with the logic $\mathbf{X}$ is the theory of Boolean algebras with an arbitrary fixed number of nullary function symbols, $\mathbf{c}_1, \ldots, \mathbf{c}_n$, and a unary function symbol, X, which denotes a Boolean endomorphism. Our main result is that the equational theory of the free algebras of this variety is decidable. In the remainder of this abstract, we will give an overview of our approach for proving this result.

Let Σ be a finite alphabet. The *de Bruijn graph* $B_d = (\Sigma^d, S_d)$ of dimension $d \geq 1$ is the graph with set of vertices Σ^d and Σ-colored edge relation defined as

$$S_d \coloneqq \{(bv, a, va) \ : \ a, b \in \Sigma, v \in \Sigma^{d-1}\} \subseteq \Sigma^d \times \Sigma \times \Sigma^d \ .$$

One may think of the de Bruijn graph as a deterministic automaton that 'remembers' the d letters that were most recently read. We define the *de Bruijn graph mapping problem* to be the decision problem that asks, for an input graph with Σ-coloring on the edges, $G = (V_G, E_G)$, whether or not there exist $d \geq 1$ and a homomorphism (i.e., colored-edge-preserving function) from B_d to G.

Theorem 1. *The unifiability problem for $\mathcal{E}_{\mathbf{X}}$ is computationally equivalent to the de Bruijn graph mapping problem.*

The proof of Theorem 1 uses Stone duality and a step-wise construction for the free $\mathcal{E}_{\mathbf{X}}$-algebra, as we will explain further below. Given this result, our new goal is to show that the de Bruijn graph mapping problem is decidable. For this, we introduce two notions on graphs, that we call *cycle-connected* and *power-connected*, and we prove:

Theorem 2. *A graph G has a cycle- and power-connected subgraph if, and only if, there exist $d \geq 1$ and a homomorphism $B_d \to G$.*

Since we also show that it can be checked (in exponential time) whether or not a graph has a cycle- and power-connected subgraph, Theorem 2 in particular implies that the de Bruijn graph mapping problem is decidable, from which the decidability of unifiability in $\mathcal{E}_{\mathbf{X}}$ then follows. Since the reduction from an instance of unifiability to an instance of de Bruijn graph mapping in Theorem 1 takes at most exponential time, our algorithm as a whole gives a 2-EXPTIME upper bound.

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To explain a bit more about the proof of Theorem 1, let us first stratify the set of $\mathcal{E}_{\mathbf{X}}$-terms by *depth*, i.e., the maximum nesting of function symbols occurring in a term. For any $d \geq 0$, there are, up to equivalence, only finitely many candidate unifiers *of depth* $\leq d$. Writing $\mathbb{A}_d$ for the set of equivalence classes of ground $\mathbf{X}$-formulas of depth $\leq d$, we thus obtain a chain of inclusions of finite sets

$$\mathbb{A}_0 \hookrightarrow \mathbb{A}_1 \hookrightarrow \mathbb{A}_2 \hookrightarrow \cdots \quad . \tag{2}$$

Each of the finite sets $\mathbb{A}_d$ carries the syntactic structure of a Boolean algebra, and, for each $d \geq 0$, a Boolean algebra homomorphism,

$$\mathrm{X}_d^{\mathbb{A}} : \mathbb{A}_d \to \mathbb{A}_{d+1} \ ,$$

which sends the equivalence class of a formula ϕ of depth $\leq d$ to the equivalence class of $\mathrm{X}\phi$. The chain (2) decomposes the initial algebra for $\mathcal{E}_{\mathbf{X}}$ as a colimit of a chain of finite algebras. Now, following a methodology pioneered in [3], we apply finite Stone duality to the diagram (2), in order to obtain a dual diagram in the category of finite sets, namely, an inverse chain

$$V_0 \twoheadleftarrow V_1 \twoheadleftarrow V_2 \twoheadleftarrow V_3 \twoheadleftarrow \cdots \tag{3}$$

where $V_d \coloneqq \mathrm{At}\,\mathbb{A}_d$, the set of *atoms* of the Boolean algebra $\mathbb{A}_d$. Using duality, we show that the operation X also gives rise to a graph structure on the set V_d, which makes it isomorphic to B_d, the de Bruijn graph of dimension d. A further application of Stone duality to the possible solutions of a unification problem then shows that a unifying substitution one-to-one corresponds to a graph homomorphism from one of the graphs V_d to a graph G, which can be computed within exponential time from the formulas to be unified.

In order to prove Theorem 2, we significantly extend a number of existing results from the literature [2, 1]. There, the restriction of the de Bruijn graph mapping problem to *deterministic* target graphs was shown decidable, by characterizing the deterministic homomorphic images of de Bruijn graphs as precisely those graphs which are strongly connected and *d-synchronizing*. The latter condition says that, for every $w \in \Sigma^d$, there is a node y_w such that for every $x \in V_G$, there exists a path $x \xrightarrow{w} y_w$ in G.

However, the homomorphic image of a deterministic graph, such as B_d, may fail to be deterministic, and Theorem 1 implies that essentially any graph, not necessarily deterministic, can occur as the graph associated with a unification problem for the logic $\mathbf{X}$. Our definitions of ‘power-connected’ and ‘cycle-connected’ capture properties that generalize the d-synchronizing condition to the non-deterministic setting, in two distinct directions. We designed the conditions in such a way that any graph admitting a homomorphism from a de Bruijn graph must satisfy both conditions. The most difficult combinatorial part of our work lies in the converse direction of Theorem 2. A crucial idea there is that of *minimizers*, which originates in the literature on string compression algorithms [6, 5]. This notion allows one, in any non-highly-periodic word $w \in \Sigma^d$ for large d, to single out a particular position in the word w which remains stable when walking from w in any direction in the de Bruijn graph B_d during at most r steps.

The work we describe here instantiates a general (co-)algebraic approach towards unification, which we plan to develop in further work. We hope that this will allow us to delineate the precise scope of the method that we followed here, addressing in particular the question of whether or not it can be helpful for the open problem of decidability of unifiability in basic modal logic $\mathbf{K}$. The analogous result to Theorem 1 for $\mathbf{K}$ was stated in [4], but the corresponding combinatorial problem on hypergraphs is currently out of our reach. Further questions for future work include whether the 2-EXPTIME upper bound on unifiability in $\mathbf{X}$ is tight, and how difficult it is to actually compute unifiers, if they exist: Our current method only gives a quadruple-exponential bound, but we expect that a more syntactic analysis of the problem could improve on this.

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