Canonical Approximations of Modal Logics

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Recall that a modal logic Λ is called *canonical* when its variety of algebras is closed under taking canonical extensions [2, Definition 5.44]. It is well known that this is equivalent to being $\mathcal{D}$-persistent, i.e. having the property that for every descriptive frame of Λ, the underlying Kripke frame is also a Λ-frame [2, Proposition 5.85].

The most important property of canonical logics is that they are *strongly Kripke complete*. As such, canonicity is a major tool for establishing Kripke completeness for modal logics. In addition, many logics of interest are canonical. Sahlqvist's completeness theorem states that every logic axiomatised by Sahlqvist formulas is canonical [2, Theorem 4.42], thus establishing a convenient syntactic description for a large subclass of canonical logics. Moreover, the Fine-van Benthem theorem states that every logic characterised by an elementary class of frames is canonical [3, Theorem 10.19].

In spite of these results, several well-known modal logics are not canonical, most notably the McKinsey logic **K.1** (or **KM**), the Gödel-Löb logic **GL** and Grzegorczyk's logic **Grz** [3, Section 6.2] [4].[1] In addition, several common extensions, such as **Grz.2** and **Grz.3**, are not canonical.

We are interested in finding closest canonical "approximations" for (non-canonical) normal modal logics.

Approximations. Let $\mathsf{NExt}(\mathbf{K})$ denote the set of all normal modal logics, and let $\mathcal{X} \subseteq \mathsf{NExt}(\mathbf{K})$ be a set of normal modal logics such that $(\mathcal{X}, \subseteq)$ forms a complete lattice. For a logic Λ not necessarily in $\mathcal{X}$, define the *$\mathcal{X}$-approximation of Λ from below* resp. *from above* to be

$$\mathcal{X}_{\uparrow}(\Lambda) \coloneqq \bigvee\{\Lambda' \in \mathcal{X} \mid \Lambda' \subseteq \Lambda\} \qquad \text{and} \qquad \mathcal{X}_{\downarrow}(\Lambda) \coloneqq \bigwedge\{\Lambda' \in \mathcal{X} \mid \Lambda \subseteq \Lambda'\}$$

respectively. Clearly, a completely analogous definition can be used in the intuitionistic setting.

When $(\mathcal{X}, \subseteq)$ is a complete *sub*lattice of $(\mathsf{NExt}(\mathbf{K}), \subseteq)$ the meet is the intersection of logics and the join is the sum, i.e. the least normal modal logic containing the union of the logics, and we obtain

$$\mathcal{X}_{\uparrow}(\Lambda) \subseteq \Lambda \subseteq \mathcal{X}_{\downarrow}(\Lambda).$$

In this case the approximation from above is the least logic in $\mathcal{X}$ extending Λ and the approximation from below is the greatest sublogic of Λ contained in $\mathcal{X}$.

Taking for $\mathcal{X}$ the set of weakly Kripke complete normal modal logics, the approximation from above is just the logic of the frame class, i.e. $\mathsf{Log}(\mathsf{Fr}(\Lambda))$. In the intuitionistic setting, [1, 5] studied approximations where the set of super-intuitionistic subframe logics and the set of super-intuitionistic stable logics are taken for $\mathcal{X}$. Canonical approximations however, have not been studied before.

[1] Recall that **K.1** is the normal modal logic axiomatised by the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$, **GL** is the logic of irreflexive conversely wellfounded frames and **Grz** the logic of reflexive conversely wellfounded frames [3, Section 3.5 and Table 4.2].

Canonical approximations. Let us write Can for the set of canonical normal modal logics. We note the following.

Theorem 1. *Can is closed under arbitrary sums and finite intersections, but not under infinite intersections. Hence it forms a complete lattice, and a sublattice of $(\mathsf{NExt}(\mathbf{K}), \subseteq)$, but not a complete sublattice.*

Even though the closure under intersections is stated as Problem 10.2 in [3], the proof turns out to be an easy exercise.[2]

Since Can is closed under arbitrary sums, $\mathsf{Can}_\uparrow(\Lambda) \subseteq \Lambda$ for every logic Λ. Interestingly, however, the dual inequality need not hold: the canonical approximation from above of a logic need not extend the logic. This is exemplified by the following theorem, which follows from the Fine-van Benthem theorem.

Theorem 2. *Let Λ be a logic that has the finite model property. Then $\mathsf{Can}_\downarrow(\Lambda) = \mathsf{Can}_\uparrow(\Lambda)$.*

Clearly this means that for non-canonical logic Λ which has the finite model property, e.g. **GL** or **Grz**, $\Lambda \not\subseteq \mathsf{Can}_\downarrow(\Lambda)$. In fact the canonical approximation from above of a logic can be expressed as a kind of special case of the one from below by the formula

$$\mathsf{Can}_\downarrow(\Lambda) = \mathsf{Can}_\uparrow\Big(\bigcap\{\Lambda' \in \mathsf{Can} \mid \Lambda \subseteq \Lambda'\}\Big).$$

Recall that over **S4**, the McKinsey axiom, denoted *.1*, corresponds to the class of frames in which every point sees a point that sees only itself. The *.2* axiom expresses the *confluence* or *Church-Rosser* property, and the *.3* axiom expresses linearity of frames [3, Section 3.5 and Table 4.2]. Using selection-based methods, we compute the canonical approximations of **Grz.2** and **Grz.3**.

Theorem 3.

(i) $\mathsf{Can}_\downarrow(\mathbf{Grz.2}) = \mathsf{Can}_\uparrow(\mathbf{Grz.2}) = \mathbf{S4.2.1}$,

(ii) $\mathsf{Can}_\downarrow(\mathbf{Grz.3}) = \mathsf{Can}_\uparrow(\mathbf{Grz.3}) = \mathbf{S4.3.1}$.

In a sense, in these two cases the canonical approximation is obtained by "just" dropping the converse wellfoundedness from the frame conditions. This raises the question whether something similar happens for other non-canonical logics, in particular **Grz** itself and the analogous extensions of **GL**.

References

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[2] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*, volume 53. Cambridge University Press, 2001. Cambridge Tracts in Theoretical Computer Science.

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[2] The fact that canonicity is not preserved under infinite intersections can be seen for example by considering the logic **GL**, known to be non-canonical, which can be shown to equal the intersection of the logics $\mathbf{K4} \oplus \Box^n \bot$.