On assume-guarantee contract algebras

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Abstract

Contract-based design has emerged as a way to design a wide variety of systems in engineering. Contracts have a legal part and a technical part. Only the technical part will be relevant in what follows. Informally, a contract is a pair (A, G), where A is the set of assumptions under which the system is assumed to operate and G is the set of guarantees that the system must provide. For examples and details see [1].

In [3], an algebraic perspective on assume-guarantee contracts is proposed. This proposal relies heavily on a construction involving Boolean algebras. Given any Boolean algebra $\mathbf{B} = (B, \cap, \cup, \ ', 0, 1)$, the set of pairs $(a, b) \in B \times B$ such that $a \cup b = 1$ is taken to be the universe of the assume-guarantee contract algebra associated to $\mathbf{B}$. Let us call $S(B)$ to this set. However, the structures thus proposed lack a clearly prescribed set of basic operations, necessary if we want to see them as a class of algebras in the sense of Universal Algebra.

Many different operations can be defined on the given set, among which we consider the following:

- $(a, b) \wedge (c, d) := (a \cap c, b \cup d)$,
- $(a, b) \vee (c, d) := (a \cup c, b \cap d)$,
- $(a, b) \cdot (c, d) := (a \cap c, (b \cap d) \cup (a \cap c)')$,
- $\sim (a, b) := (b, a)$,
- $\perp := (0, 1)$,
- $\top := (1, 0)$,
- $e := (1, 1)$.

If we take them as basic operations, it is possible to define a class of algebras of type (2,2,2,1,0,0,0).

In our talk, using well-known results from the literature (see, for example, [5] and [7]), we manage to describe these algebras as members of the subvariety of bounded odd Sugihara monoids generated by the three-element chain, that is, the variety of bounded three-valued Sugihara monoids. Furthermore, any bounded three-valued Sugihara monoid is isomorphic (as a bounded Sugihara monoid) to an assume-guarantee contract algebra. That is to say, the class of assume-guarantee contract algebras generates the variety of bounded three-valued Sugihara monoids.

As a consequence of the aforementioned facts, we get the following result, where BA and B3SM are the categories of Boolean algebras and bounded three-valued Sugihara monoids, respectively.

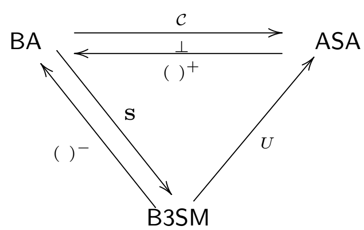
Proposition: *Functors* $(\ )^- : \mathsf{B3SM} \rightleftarrows \mathsf{BA} : S$ *witness a categorical equivalence, where* S *is the functor induced by the construction defined above and* $(\ )^-$ *is the functor induced by taking the subalgebra formed by the elements below the identity* e.

Furthermore, we consider other well-studied varieties that are term equivalent to the mentioned variety of bounded three-valued Sugihara monoids and hence, that provide

alternative abstract characterizations of assume-guarantee contract algebras in alternative signatures. More concretely, we show that assume-guarantee contract algebras may be regarded either as elements of the variety of centred three-valued double p-algebras (see [2]) or as elements of the variety of centred three-valued Lukasiewicz algebras (see [6]).

In [4], the author finds an adjunction between the category of Boolean algebras and the category ASA of Stone algebras (Heyting algebras satisfying the equation $\neg x \vee \neg\neg x = 1$) expanded with a constant e satisfying the identity $e \to x = \neg\neg x$. He takes the set $C(B)$ of contracts on a Boolean algebra $\mathbf{B}$ as an algebra in ASA with $e = (1, 1)$, not in B3SM. The assignment $\mathbf{B} \mapsto \mathbf{C}(\mathbf{B})$ defines a functor $\mathbf{C} : \mathsf{BA} \to \mathsf{ASA}$. It is shown that $\mathbf{C}$ is part of an adjoint pair $\mathbf{C} \dashv \mathbf{Clos}$, where $\mathbf{Clos}(\mathbf{A})$ is the Stonean subalgebra of an algebra $\mathbf{A}$ in ASA formed by its complemented elements. In any pseudo-complemented bounded distributive lattice $(A; \wedge, \vee, \neg, 0, 1)$ having e as minimum dense element, the sublattice $[e) = \{a \in A : a \leq e\}$, together with the unary operation N defined by $Na := \neg a \wedge e$, is a Boolean lattice isomorphic to $\mathbf{Clos}(A)$. Instead of $\mathbf{Clos}$, the author could have taken the functor $(\ )^+ : \mathsf{ASA} \to \mathsf{BA}$ defined by the assignment $A \mapsto [e)$. Clearly, we also have that $\mathbf{C} \dashv (\ )^+$.

Due to the functional completeness of bounded three-valued Sugihara monoids, it follows that, given a Boolean algebra $\mathbf{B}$, the algebra $\mathbf{S}(\mathbf{B})$ has the underlying structure of a Heyting algebra, which is Stonean and has e as minimum dense element. As a consequence, we have a forgetful functor $\mathbf{U} : \mathsf{B3SM} \to \mathsf{ASA}$ making the following diagram commute.



Since $\mathbf{S}$ and $(\ )^-$ witness an equivalence, it follows that $(\ )^- \dashv \mathbf{S}$ and, in consequence, $\mathbf{U} = \mathbf{C} \circ (\ )^- \dashv \mathbf{S} \circ (\ )^+ \cong \mathbf{S} \circ \mathbf{Clos}$. This establishes a relation between our results and those in [4].

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