Implication free reduct of intuitionism, or p-algebras revisited

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Abstract

We give a new description of free (distributive) p-algebras, which in particular yields a normal form theorem for terms. We also prove some new results about the subquasivariety lattice, which for lack of space we can only signal below.

1 Introduction

A *distributive p-algebra* (from now on, simply, a *p-algebra*) is an algebra $(A; \wedge, \vee, ^*, 0, 1)$ where $(A; \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and the unary operation * satisfies the equivalence

$$x \wedge y = 0 \iff x \leq y^*.$$

P-algebras are a variety, Pa, consisting of term-subreducts of Heyting algebras, without implication but with the term $x^* := x \to 0$. Their subvariety lattice is a chain of type $\omega + 1$:

$$\mathsf{Pa}_{-1} \subset \mathsf{Pa}_0 \subset \mathsf{Pa}_1 \subset \cdots \subset \mathsf{Pa}$$

where Pa_{-1} is the trivial variety, and Pa_k is generated by a single subdirectly irreducible algebra. In contrast to Heyting algebras, they are not 1-regular and not even 1-subtractive, although they are 0-subtractive (yet still not 0-regular).

They were studied extensively in 1970s and 1080s, and then the research petered out. We try to rekindle the interest in these algebras providing a new description of free p-algebras, based entirely on a thorough understanding of completely meet-irreducible congruences. This enables us to formulate a normal form theorem for p-algebra terms, and prove some results about the lattice of subqusivarieties of p-algebras, sharpening the existing ones.

2 Free algebras

We build free algebras using completely meet-irreducible congruences. The most important observation on them is that they come in two layers, given in the following definition, where $\mathbf{Cm}\,\mathbf{A}$ stands for the set of completely meet-irreducible congruences of $\mathbf{A}$; for $\mu \in \mathbf{Cm}\,\mathbf{A}$, we write μ^+ for the unique cover of μ in the lattice $\mathbf{Con}\,\mathbf{A}$ of all congruences of $\mathbf{A}$, and $M(\alpha)$ is $\{\mu \in \mathbf{Cm}\,\mathbf{A} : \alpha \subseteq \mu\}$.

Definition 2.1. *Let* $\mathbf{A} \in \mathsf{Pa}$. *Put*

$$\mathrm{I}_{\mathbf{A}} := \{\mu \in \mathbf{Cm}\,\mathbf{A} : \mathbf{A}/\mu \cong \overline{\mathbf{B}}_0\} = \{\mu \in \mathbf{Cm}\,\mathbf{A} : \mu^+ = \mathbf{1}^{\mathbf{A}}\},$$
$$\mathrm{II}_{\mathbf{A}} := \{\mu \in \mathbf{Cm}\,\mathbf{A} : \mathbf{A}/\mu \cong \overline{\mathbf{B}}_n \textit{ for } n > 0\} = \{\mu \in \mathbf{Cm}\,\mathbf{A} : M(\mu^+) \subseteq \mathrm{I}_{\mathbf{A}}\}.$$

Let $\mathbf{F}_n(k)$ be the free k-generated p-algebra in the variety Pa_n. Let $T \in \mathcal{P}(k)$, define $f_T : \{x_1, \dots, x_k\} \to \{0,1\}$ putting $f_T(x_i) := 1$ if $i \in T$, and 0 otherwise. Let $\bar{f}_T$ be the homomorphism onto $\mathbf{2}$ extending f_T. For any $T \in \mathcal{P}(k)$ and any $\mu \in \mathbf{Cm}\,\mathbf{F}_n(k)$ we have $\mu \in \mathrm{I}_{\mathbf{F}_n(k)} \iff \mu = \ker \bar{f}_T$. Next, define

$$x_T := \bigwedge_{i \in T} x_i \wedge \bigwedge_{i \notin T} x_i^*. \tag{at}$$

Then, for any $T \in \mathcal{P}(k)$, the element x_T is an atom and every atom of $\mathbf{F}_n(k)$ is of this form. Therefore, if $\mu \in \mathrm{I}_{\mathbf{F}_n(k)}$ then $1/\mu = [x_T)$ for some $T \in \mathcal{P}(k)$. Write μ_T for that μ.

It can be shown that each join-irreducible element $p \in \mathbf{F}_n(k)$ is the smallest element of $1/\mu$ for some $\mu \in \mathbf{Cm}\,\mathbf{F}_n(k)$. For an arbitrary but fixed $\mu \in \mathbf{Cm}\,\mathbf{F}_n(k)$, we define

$$L := \{i < k : x_i \in 1/\mu\}, \qquad \mathcal{T} := \{T \in \mathcal{P}(k) : \mu \subseteq \mu_T\}, \qquad p^L_{\mathcal{T}} := \Big(\bigvee_{T \in \mathcal{T}} x_T\Big)^{**} \wedge \bigwedge_{i \in L} x_i.$$

Intuitively, L encodes the set of generators that μ maps to 1, and $\mathcal{T}$ encodes the set of maximal congruences extending μ. For any $L \subseteq k$ and nonempty $\mathcal{T} \subseteq \mathcal{P}(k)$, such that $L \subseteq \bigcap \mathcal{T}$, we will write $\mu^L_{\mathcal{T}}$ for the unique congruence in $\mathbf{Cm}\,\mathbf{F}_n(k)$ such that $1/\mu^L_{\mathcal{T}} = [p^L_{\mathcal{T}})$.

Definition 2.2. *Let $\mathcal{T}$ and $\mathcal{S}$ be nonempty subsets of $\mathcal{P}(k)$. Let $L \subseteq \bigcap \mathcal{T}$ and $K \subseteq \bigcap \mathcal{S}$. Define an ordering relation $\leq^{\mathbf{Cm}}$ on $\mathbf{Cm}\,\mathbf{F}_n(k)$ putting*

$$\mu^L_{\mathcal{T}} \leq^{\mathbf{Cm}} \mu^K_{\mathcal{S}} \iff \mathcal{S} \subseteq \mathcal{T} \text{ and } L \subseteq K.$$

Theorem 2.3 (Structure of free p-algebra)**.** *We have:*

$$\mathbf{F}_n(k) \cong \mathrm{Up}(\mathbf{Cm}\,\mathbf{F}_n(k), \leq^{\mathbf{Cm}})$$

where Up *is the usual up-set operator.*

Theorem 2.4 (Normal form theorem)**.** *Every element t of the algebra $\mathbf{F}_n(k)$ is of the form*

$$t = \bigvee \max\{p^L_{\mathcal{T}} \in \mathcal{J}(\mathbf{F}_n(k)) : p^L_{\mathcal{T}} \leq t\}$$

where $\mathcal{J}(-)$ stands for the set of join-irreducible elements.

For $n \geq \mathcal{P}(k)$, this yields $|\mathcal{J}(\mathbf{F}_n(k))| = \sum_{i=0}^{k} \binom{k}{i}(2^{2^i} - 1)$, a formula known before, but our calculation is much easier.

3 Subquasivarieties

Using our description of free algebras and a few tricks we can show that

- Each free p-algebra belongs to the splitting companion (in the quasivariety lattice) of Pa_3. For $n \geq 2$, the interval $[\mathsf{Pa}_n, \mathsf{Pa}_{n+1}]$ is of cardinality continuum.
- For $n \geq 3$, the variety Pa_n is not structurally complete in the algebraic sense, in spite of the fact that the corresponding logic is structurally complete by a result of G. Mints. The discrepancy is due to non-algebraizability of the logic.