Finitely Generated Varieties of Commutative BCK-algebras: Covers

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BCK-algebras were first introduced in [3] as an algebraic semantics for non-classical logic that uses only implication. Every BCK-algebra admits an ordering, and if it satisfies commutativity law (which in BCK-algebras is not the same as the standard commutativity of binary operation), then the underlying poset is a meet semi-lattice. For the sake of brevity, we will refer to commutative BCK-algebras as "cBCK-algebras". Unlike BCK-algebras, cBCK-algebras form a variety.

The variety of all cBCK-algebras has several noteworthy properties, including congruence distributivity and 3-permutability. In contrast, no nontrivial subvariety is 2-permutable. That is important since having arithmetical variety would facilitate the investigation. Finitely generated varieties of cBCK-algebras are semisimple, i.e. any subdirectly irreducible member is simple. Also, every finite simple cBCK-algebra is hereditary simple. A crucial fact is that subdirectly irreducible cBCK-algebras are (regarding their order) rooted trees [5], [2].

We are interested in covers of finitely generated varieties of cBCK-algebras. Let $\mathcal{V}$ be a finitely generated variety of cBCK-algebras. Then, there exist $\mathbf{A}_1, \ldots, \mathbf{A}_n$ finite subdirectly irreducible cBCK-algebras such that $\mathcal{V} = \mathrm{V}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. From congruence distributivity, it follows that $\mathcal{V} = \mathrm{V}(\mathbf{A}_1) \vee \cdots \vee \mathrm{V}(\mathbf{A}_n)$. Therefore, investigating covers of finitely generated varieties can be reduced to investigating covers of varieties generated by a single finite subdirectly irreducible cBCK-algebra. From now on, let $\mathcal{V} = \mathrm{V}(\mathbf{A})$, where $\mathbf{A}$ is finite simple subdirectly irreducible cBCK-algebra. An important observation is that $\mathrm{Si}(\mathcal{V})$ (subdirectly irreducible members of $\mathcal{V}$) consists (up to isomorphisms) only of $\mathrm{S}(\mathbf{A})$ (subalgebras of $\mathbf{A}$).

The fact that $\mathrm{Si}(\mathcal{V}) = \mathrm{S}(\mathbf{A})$ motivates us to first explore $\mathrm{S}(\mathbf{A})$. There are two kinds of subalgebras: downsets and the others. The others can be characterised as a set of elements of $\mathbf{A}$ that have height divisible by some integer $k > 1$. Under some conditions, such a set indeed forms a subalgebra. The detailed characterisation is the subject of the first part of the talk.

The second part of the presentation focuses on the covers. The goal is to find all covers of $\mathcal{V}$, i.e. to find subdirectly irreducible cBCK-algebras that generate the covers. The construction involves considering all subalgebras of $\mathbf{A}$ and then considering their extensions by adding a leaf to some vertex (not the root). We prove that by the construction, we obtain a cover and that every cover is achievable by the construction.

References

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