Equivalential Algebras With Conjunction on Dense Elements

Sławomir Przybyło

Institute of Mathematics, University of the National Education Commission, Krakow, Poland

Abstract

We study the variety generated by the three-element equivalential algebra with conjunction on the dense elements. We prove the representation theorem which let us construct the free algebras in this variety. Next, we compute the formula for the cardinality of these algebras.

1 Introduction

According to [2], there are only finitely many polynomial clones on a finite algebra which generate congruence permutable Fregean varieties. A variety $\mathcal{V}$ with a distinguished constant term 1 is called **Fregean** if every algebra $\mathbf{A} \in \mathcal{V}$ is: **1-regular**, (i. e., $1/\alpha = 1/\beta$ implies $\alpha = \beta$ for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$) and **congruence orderable** (i. e., $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$ implies $a = b$ for all $a, b \in A$) [2, p. 597].

If a three-element algebra $\mathbf{A}$ generates a congruence permutable Fregean variety, then the lattice of congruences on $\mathbf{A}$ is a three-element chain. By [2, Corollary 2.8], due to the behavior of the commutator operation on a three-element algebra, we can distinguish four polynomially nonequivalent algebras, that generate congruence permutable Fregean varieties.

Two of them are well known: the three-element equivalential algebra and the three-element Brouwerian semilattice. The equivalential algebras are solvable, so they are of type 2 ([2, p. 606]) in the sense of Tame Congruence Theory of Hobby and McKenzie [1]. However, the Brouwerian semilattices are congruence distributive and so they are of type 3.

In the other two cases we are dealing with a mixed type. In the first case, we have type 3 at the top of congruence lattice and type 2 at its bottom. An example of algebra, which meets these conditions is the three-element equivalential algebra with conjunction on the regular elements. The variety generated by this algebra was investigated in [3].

2 Main results

The aim of this talk is present recent results on the variety generated by the three-element algebra, in which the commutator operation behaves in the opposite way: type 2 is at the top of congruence lattice and type 3 at its bottom (most of these results can be found in the article [4], written with Katarzyna Słomczyńska). Such structure is the subreduct of the three-element Heyting algebra, with the equivalence operation and the second binary operation which is conjunction on the dense elements.

Definition 1. An **equivalential algebra with conjunction on the dense elements** is an algebra $\mathbf{D} := (\{0, *, 1\}, \cdot, d, 1)$ of type $(2, 2, 0)$, which is the reduct of the three-element Heyting algebra $\mathbf{H} = (\{0, *, 1\}, \wedge, \vee, \to, 0, 1)$ with an order: $0 < * < 1$, the constant 1, the equivalence operation $\cdot$ such that $x \cdot y := (x \to y) \wedge (y \to x)$ (we adopt the convention of associating to the left and ignoring the symbol of equivalence operation), and an additional binary operation d such that: $d(x, y) := x00x \wedge y00y$.

We denote by $\mathcal{V}(\mathbf{D})$ the variety generated by $\mathbf{D}$. A crucial role in the construction of the finitely generated free algebras is played by the subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$.

Proposition 2. *There are only three (up to isomorphism) nontrivial subdirectly irreducible algebras in* $\mathcal{V}(\mathbf{D}) : \mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}$, *where:*

$$\mathbf{2} := \{\{0,1\}, \cdot, d, 1\}, \text{ where } d \equiv 1,$$
$$\mathbf{2}^{\wedge} := \{\{*,1\}, \cdot, d, 1\}, \text{ where } d(x,y) := x \wedge y.$$

Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. We denote by $\mathrm{Cm}(\mathbf{A})$ the set of all completely meet-irreducible congruences on $\mathbf{A}$ and we use the following notation:

$$\overline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}\},$$
$$\underline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{D}\},$$
$$P := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}^{\wedge}\},$$
$$L := \overline{L} \cup \underline{L}.$$

To construct the free algebras in $\mathcal{V}(\mathbf{D})$ we need the notion of the hereditary sets.

Definition 3. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $Z \subseteq \mathrm{Cm}(\mathbf{A})$. A set Z is **hereditary** if:

1. $Z = Z\uparrow$,
2. $\overline{L} \subseteq Z$ or $((\overline{L} \cap Z) \cup \{\mathbf{1_A}\}, \bullet)$ is a hyperplane in $(\overline{L} \cup \{\mathbf{1_A}\}, \bullet)$, where $\mu_1 \bullet \mu_2 := (\mu_1 \div \mu_2)'$ for $\mu_1, \mu_2 \in \overline{L}$ ($\div$ denotes the symmetric difference)

We will denote by $\mathcal{H}(\mathbf{A})$ the set of all hereditary subsets of $\mathrm{Cm}(\mathbf{A})$.

Now we give our main result, i.e. the representation theorem:

Theorem 4. *Let* $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ *and let* $\mathbf{A}$ *be finite. Then the map* $M : A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\}$ *is the isomorphism between* $\mathbf{A}$ *and* $(\mathcal{H}(\mathbf{A}), \leftrightarrow, \boldsymbol{d}, \boldsymbol{1})$, *where*

$$Z \leftrightarrow Y := ((Z \div Y)\downarrow)'$$
$$\boldsymbol{d}(Z,Y) := [Z \cup ((Z\downarrow)' \cap L)] \cap [Y \cup ((Y\downarrow)' \cap L)],$$
$$\boldsymbol{1} := \mathrm{Cm}(A),$$

for $Z, Y \in \mathcal{H}(\mathbf{A})$.

Using this theorem, we can construct the finitely generated free algebras in $\mathcal{V}(\mathbf{D})$, which we denote by $\mathbf{F_D}(n)$, and find the formula for the cardinality of these algebras:

$$|\mathbf{F_D}(n)| = 2^{3^n - 2^n} + \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}.$$

References

[1] Hobby D., McKenzie R., **The structure of finite algebras**, *Contemporary Mathematics*, vol. 76, American Mathematical Society, Providence, RI, 1988.

[2] Idziak P. M., Słomczyńska K., Wroński A., **Fregean Varieties**, *Int J. Algebra Comput.* 19 (2009), 595-645.

[3] Przybyło S., **Equivalential algebras with conjunction on the regular elements**, *Ann. Univ. Paedagog. Crac. Stud. Math.* 20 (2021), 63-75.

[4] Przybyło S., Słomczyńska K., **Equivalential algebras with conjunction on dense elements**, *Bulletin of the Section of Logic* 51/4 (2022), 535-554.