Intuitionistic modal logics: a minimal setting

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Abstract

We make a clean sweep of the tradition in intuitionistic modal logics by considering a new truth condition of $\Diamond$-formulas saying that in model $(W,\leq,R,V)$, $\Diamond A$ holds at $s{\in}W$ if there exists $t{\in}W$ where A holds and such that $s{\geq}{\circ}Rt$. While keeping the truth condition of $\Box$-formulas that is commonly used, we axiomatize validity in the class of all models. The resulting logic is the intuitionistic modal logic that we want to put forward as a candidate for the title of "minimal intuitionistic modal logic".

1 Syntax and semantics

Let **At** be a set of *atoms* (p, q, etc). The set **Fo** of all *formulas* (A, B, etc) is defined by $A ::= p|(A{\rightarrow}A)|\top|\bot|(A{\wedge}A)|(A{\vee}A)|\Box A|\Diamond A$. For all $A{\in}\mathbf{Fo}$, $\neg A$ is the abbreviation for $(A{\rightarrow}\bot)$.

A *Kripke frame* or a *KF* is a structure of the form $(W,\leq,R)$ where W is a nonempty set, $\leq$ is a partial order on W and R is a binary relation on W. Let $\mathcal{C}^{\mathbf{kf}}_{\mathbf{all}}$ be the class of all KFs. A KF $(W,\leq,R)$ is *forward (respectively: backward; downward) confluent* if for all $s,t{\in}W$, if $s{\geq}{\circ}Rt$ then $sR{\circ}{\geq}t$ (respectively: for all $s,t{\in}W$, if $sR{\circ}{\leq}t$ then $s{\leq}{\circ}Rt$; for all $s,t{\in}W$, if $s{\leq}{\circ}Rt$ then $sR{\circ}{\leq}t$). Let $\mathcal{C}^{\mathbf{kf}}_{\mathbf{fc}}$ (respectively: $\mathcal{C}^{\mathbf{kf}}_{\mathbf{bc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{dc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{fbc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{fdc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{bdc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{fbdc}}$) be the class of all forward (respectively: backward; downward; forward and backward; forward and downward; backward and downward; forward, backward and downward) confluent KFs. A *valuation on a KF* $(W,\leq,R)$ is a function $V : \mathbf{At} \longrightarrow \wp(W)$ associating a $\leq$-closed subset of W to each atom. Such a function can be extended as a function $V : \mathbf{Fo} \longrightarrow \wp(W)$ associating to each $A{\in}\mathbf{Fo}$ a $\leq$-closed subset $V(A)$ of W defined as usual when either A is an atom, or the main connective of A is intuitionistic and as follows otherwise: (**i**) $V(\Box A){=}\{s{\in}W$: for all $t{\in}W$, if $s{\leq}{\circ}Rt$ then $t{\in}V(A)\}$; (**ii**) $V(\Diamond A){=}\{s{\in}W$: there exists $t{\in}W$ such that $s{\geq}{\circ}Rt$ and $t{\in}V(A)\}$. A *relational model* is a couple consisting of a KF and a valuation on that KF. *Truth in a relational model, validity in a KF* and *validity on a class of KFs* are defined as usual. For all classes $\mathcal{C}$ of KFs, let $\mathtt{Log}(\mathcal{C})$ be the *logic of* $\mathcal{C}$.

A *H-modal algebra* or a *HMA* is a structure of the form $(H,\leq_H,\rightarrow_H,\Box_H,\Diamond_H)$ where $(H,\leq_H,\rightarrow_H)$ is a Heyting algebra and $\Box_H : H{\longrightarrow}H$ and $\Diamond_H : H{\longrightarrow}H$ are operators such that for all $a,b,c{\in}H$: (**i**) $\Box_H\top_H{=}\top_H$; (**ii**) $\Box_H(a{\wedge}_Hb){=}\Box_Ha{\wedge}_H\Box_Hb$; (**iii**) $\Diamond_H\bot_H{=}\bot_H$; (**iv**) $\Diamond_H(a{\vee}_Hb){=}\Diamond_Ha{\vee}_H\Diamond_Hb$; (**v**) if $\Diamond_Ha{\leq}_Hb{\vee}_H\Box_H(a{\rightarrow}_Hc)$ then $\Diamond_Ha{\leq}_Hb{\vee}_H\Diamond_Hc$. Let $\mathcal{C}^{\mathbf{hma}}_{\mathbf{all}}$ be the class of all HMAs. A HMA $(H,\leq_H,\rightarrow_H,\Box_H,\Diamond_H)$ is *forward (respectively: backward; downward) confluent* if for all $a,b{\in}H$, $\Diamond_H(a{\rightarrow}_Hb){\leq}_H(\Box_Ha{\rightarrow}_H\Diamond_Hb)$ (respectively: $(\Diamond_Ha{\rightarrow}_H\Box_Hb){\leq}_H\Box_H(a{\rightarrow}_Hb)$; $\Box_H(a{\vee}_Hb){\leq}_H\Diamond_Ha{\vee}_H\Box_Hb$). Let $\mathcal{C}^{\mathbf{hma}}_{\mathbf{fc}}$ (respectively: $\mathcal{C}^{\mathbf{hma}}_{\mathbf{bc}}$;

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$\mathcal{C}^{\mathbf{hma}}_{\mathbf{dc}}$; $\mathcal{C}^{\mathbf{hma}}_{\mathbf{fbc}}$; $\mathcal{C}^{\mathbf{hma}}_{\mathbf{fdc}}$; $\mathcal{C}^{\mathbf{hma}}_{\mathbf{bdc}}$; $\mathcal{C}^{\mathbf{hma}}_{\mathbf{fbdc}}$) be the class of all forward (respectively: backward; downward; forward and backward; forward and downward; backward and downward; forward, backward and downward) confluent HMA. A *valuation on a HMA* $(H, \leq_H, \rightarrow_H, \Box_H, \Diamond_H)$ is a function $V : \mathbf{At} \longrightarrow H$ associating an element of H to each atom. Such a function can be extended as a function $V : \mathbf{Fo} \longrightarrow H$ associating to each $A \in \mathbf{Fo}$ an element $V(A)$ of H defined as usual when either A is an atom, or the main connective of A is intuitionistic and as follows otherwise: (**i**) $V(\Box A) = \Box_H V(A)$; (**ii**) $V(\Diamond A) = \Diamond_H V(A)$. An *algebraic model* is a couple consisting of a HMA and a valuation on that HMA. *Truth in an algebraic model, validity in a HMA* and *validity on a class of HMAs* are defined as usual. For all classes $\mathcal{C}$ of HMAs, let $\mathtt{Log}(\mathcal{C})$ be the *logic of* $\mathcal{C}$.

2 Axiomatization and completeness

An *intuitionistic modal logic* is a set of formulas closed for uniform substitution, containing the standard axioms of **IPL**, closed with respect to the standard inference rules of **IPL**, containing the axioms $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, $\Box(p \vee q) \rightarrow ((\Diamond p \rightarrow \Box q) \rightarrow \Box q)$, $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ and $\neg \Diamond \bot$ and closed with respect to the inference rules $\frac{p}{\Box p}$, $\frac{p \leftrightarrow q}{\Diamond p \leftrightarrow \Diamond q}$ and $\frac{\Diamond p \rightarrow q \vee \Box(p \rightarrow r)}{\Diamond p \rightarrow q \vee \Diamond r}$. We also consider the axioms (**Af**) $\Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$, (**Ab**) $(\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$ and (**Ad**) $\Box(p \vee q) \rightarrow \Diamond p \vee \Box q$. Let $\mathbf{L}_{\min}$ be the least intuitionistic modal logic. For all intuitionistic modal logics $\mathbf{L}$ and for all $A \in \mathbf{Fo}$, let $\mathbf{L} \oplus A$ be the least intuitionistic modal logic containing $\mathbf{L}$ and A. Let $\mathbf{L}_{\mathbf{fc}}$ (respectively: $\mathbf{L}_{\mathbf{bc}}$; $\mathbf{L}_{\mathbf{dc}}$; $\mathbf{L}_{\mathbf{fbc}}$; $\mathbf{L}_{\mathbf{fdc}}$; $\mathbf{L}_{\mathbf{bdc}}$; $\mathbf{L}_{\mathbf{fbdc}}$) be $\mathbf{L}_{\min} \oplus (\mathbf{Af})$ (respectively: $\mathbf{L}_{\min} \oplus (\mathbf{Ab})$; $\mathbf{L}_{\min} \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ab})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Ab}) \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ab}) \oplus (\mathbf{Ad})$).

Proposition 1.

- $\mathbf{L}_{\min} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{all}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{all}})$;
- $\mathbf{L}_{\mathbf{fc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{fc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{fc}})$; $\mathbf{L}_{\mathbf{bc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{bc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{bc}})$; $\mathbf{L}_{\mathbf{dc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{dc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{dc}})$;
- $\mathbf{L}_{\mathbf{fbc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{fbc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{fbc}})$; $\mathbf{L}_{\mathbf{fdc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{fdc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{fdc}})$; $\mathbf{L}_{\mathbf{bdc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{bdc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{bdc}})$;
- $\mathbf{L}_{\mathbf{fbdc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{fbdc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{fbdc}})$.

Proposition 2.

- **WK** *[3] and* $\mathbf{L}_{\min}$ *are not comparable;*
- **WK** *[3] is strictly contained in* $\mathbf{L}_{\mathbf{fc}}$*;*
- $\mathbf{L}_{\mathbf{fc}}$ *and* **FIK** *[1] are equal;*
- $\mathbf{L}_{\mathbf{fbc}}$ *and* **IK** *[2] are equal;*
- $\mathbf{L}_{\mathbf{fbdc}}$ *is strictly contained in* **K** *— the least normal modal logic.*

All in all, $\mathbf{L}_{\min}$ is the intuitionistic modal logic that we want to put forward as a candidate for the title of "minimal intuitionistic modal logic".

References

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