Modal completeness for general scattered spaces

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Modal logic admits two, by now classical, topological semantics. One is given by interpreting the modal diamond $\Diamond$ as the closure, and the other by interpreting $\Diamond$ as the derived set[1]. We refer to [2] for a thorough overview of these semantics. A celebrated result for the closure semantics is the McKinsey and Tarski theorem stating that the modal logic $\mathsf{S4}$ is sound and complete with respect to any dense-in-itself metrizable space, in particular, any Euclidean space [9]. A landmark result for the derivative semantics is the Abashidze-Blass theorem stating that the modal logic $\mathsf{GL} = \Box(\Box p \to p) \to \Box p$ is sound and complete with respect to any ordinal $\alpha \geq \omega^\omega$ with the standard interval topology [1, 5] (See also: [4]). Earlier Esakia [7] showed that GL is sound and complete with respect to the class of scattered spaces. Recall that a topological space is *scattered* if its every non-empty subset contains a point isolated in that subset. It is easy to verify that each ordinal is a scattered space with respect to the order topology.

In modal logic, general Kripke frames constitute an important generalization of Kripke semantics. A *general Kripke frame* is a triple (X, R, A), where $A \subseteq \mathcal{P}(X)$ is a modal subalgebra of the powerset algebra. In a general frame formulas are evaluated in the algebra A. A Kripke frame can be seen as a general frame, where $A = \mathcal{P}(X)$. It is well known that, unlike Kripke semantics, every modal logic is sound and complete with respect to its general Kripke frames [6].

Similarly to Kripke frames one can consider general topological spaces for both the closure and derived set semantics. A general (topological) c-space is a pair (X, A), where X is a topological space and A is a modal subalgebra of $(\mathcal{P}(X), c)$, where $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is the topological closure. Like in general Kripke frames, in general c-spaces, formulas are evaluated in the algebra A. Bezhanishvili et al. [3] show that the McKinsey and Tarski theorem can be extended to all connected extensions of $\mathsf{S4}$ by considering general c-spaces. In particular, they showed that for every extension $L \supseteq \mathsf{S4}$, such that L is the logic of a connected $\mathsf{S4}$-algebra, there is a general c-space $(\mathbb{R}, A)$ over the real line $\mathbb{R}$ such that L is sound and complete for $(\mathbb{R}, A)$.

General topological spaces for the derived set semantics have been considered in [8] for provability logics with countably many modal operators and more recently in [10] where it was shown that the bimodal provability logic GLB is sound and complete with respect to general bi-topological spaces.

In this abstract we combine these two approaches. We will consider general topological spaces for the derived set semantics over ordinal spaces and we will prove a generalization of the Abashidze-Blass theorem for these spaces, in the same way [3] proved a generalized version of the McKinsey and Tarski theorem for general spaces over the real line.

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[1] Recall that the derived set $d(U)$ of U consists of all those points x such that every open neighbourhood of x intersects $U \setminus \{x\}$.

Definition 1. *A general d-space is a pair (X, A) with X a topological space and $A \subseteq \mathcal{P}(X)$ a modal subalgebra of $(\mathcal{P}(X), d)$, where $d : \mathcal{P}(X) \to \mathcal{P}(X)$ is the derived set operator.*

A valuation in a general d-space is a map from propositional variables to A, which is extended to all formulas in a standard way, mapping $\Diamond\varphi$ to $d[\![\varphi]\!]$. Note that every general d-space has a least subalgebra, the d-algebra generated by $\varnothing$. We call a general d-space (X, A) where X is a scattered space and A is the least d-subalgebra of $(\mathcal{P}(X), d)$, a *least scattered d-space*. Recall that $\mathsf{GL.3} = \mathsf{GL} + (\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p))$.

Theorem 1. *Let (X, A) be a least scattered d-space. Then (X, A) validates $\mathsf{GL.3}$.*

Recall that Kripke frames of $\mathsf{GL.3}$ are linear dually well-founded frames (i.e., linear GL-frames)[6]. The above result can be extended to a completeness of all extensions of $\mathsf{GL.3}$.

Theorem 2. *For every extension $L \supseteq \mathsf{GL.3}$ there exists an ordinal $\alpha \leq \omega^\omega$ and a least scattered d-space (α, A) over α such that L is the logic of (α, A).*

The above theorem can in fact be generalized to a much larger class.

Theorem 3. *Let $L \supseteq \mathsf{GL}$ be a Kripke complete extension of GL. Then there exists a countable ordinal α and a general scattered d-space (α, A) over α, such that L is the logic of (α, A). Furthermore, if L enjoys the finite model property, then $\alpha \leq \omega^\omega$.*

We leave it as an open problem whether any extension of GL (i.e., not Kripke complete ones) is complete with respect to a class of general scattered d-spaces. Another interesting direction for future research is to study least general d-spaces beyond scattered spaces and to investigate completeness of modal logics, not necessarily of extensions of GL, with respect to general topological d-spaces.

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