The algebras of Lewis's counterfactuals and their duality theory

Giuliano Rosella[1] and Sara Ugolini[2]

[1] Center for Logic, Language, & Cognition (LLC), Department of Philosophy and Education Sciences, University of Turin
giuliano.rosella@unito.it

[2] Artificial Intelligence Research Institute (IIIA - CISC), Campus de la UAB, Carrer de Can Planas, Bellaterra (Barcelona), Spain
sara@iiia.csic.es

A counterfactual conditional (or simply a counterfactual) is a conditional statement of the form "If *antecedent* were the case, then *consequent* would be the case", formalized as "$\varphi \Box\!\!\rightarrow \psi$" where the antecedent is usually assumed to be false. Counterfactuals have been studied in different fields, such as linguistics, artificial intelligence, and philosophy. The logical analysis of counterfactuals is rooted in the work of Lewis [4, 3] and Stalnaker [8] who have introduced what has become the standard semantics for counterfactual conditionals based on particular Kripke models (called *sphere models*) equipped with a similarity relation among the possible worlds. Lewis [4] develops a hierarchy of logics meant to deal with different kinds of counterfactual conditionals; they are usually referred to as *variably strict conditional logics*.

Although the research on Lewis's conditional logics has been and still is very prolific, the algebraic perspective is essentially lacking; while a few works present a semantics in terms of algebraic structures for Lewis's conditional logics ([5, 7]), the results therein are either partial or fall outside the framework of the abstract algebraic analysis. A foundational work that carries Lewis's hierarchy within the realm of the well-developed discipline of (abstract) algebraic logic is notably missing in the literature; the present contribution aims at filling this void.

To this end, we start by considering Lewis's logics as consequence relations, instead of just sets of theorems, and we introduce novel (and simpler) axiomatizations. This brings us to consider two different kinds of derivation, depending on whether the deductive rules are applied only to theorems (giving a relatively weaker calculus) or to all derivations (i.e. yielding a stronger calculus); this distinction, although relevant, is often blurred in the literature. As it is the case for modal logic (see [1, 9]), these two choices turn out to correspond to considering two different consequence relations on the intended sphere models: a *local* and a *global* one; the latter, to the best of our knowledge, has not been considered in the literature.

Inspired by some results connecting modal operators and Lewis counterfactuals (see [4]), our work unveils a deep relationship between Lewis's logics and modal logic. Specifically, we demonstrate how several model-theoretic techniques commonly used in standard Kripke semantics for modal logic (such as the generated sub-model construction) can be successfully applied to Lewis's sphere semantics, thanks to a modal operator $\Box$ that can be term-defined in the language. This allows us to, for example, prove a deduction theorem for the strong calculus (whereas the weak calculus is known to have the classical deduction theorem) and to characterize the global consequence relation in terms of the local one, paralleling analogous well-known results in modal logic (see [9]).

Furthermore, we introduce a new variety of algebras, that we call V-algebras, consisting of Boolean algebras equipped with a binary operator $\Box\!\!\rightarrow$ that stands for the counterfactual conditional. We show that the stronger calculi, associated to the global consequence relation, are strongly algebraizable in the sense of Blok-Pigozzi, with respect to (subvarieties of) V-algebras. In turn, we demonstrate that the weaker calculi, associated to the local consequence relation, are not algebraizable in general, but they correspond to the logics *preserving the degrees of truth* of the same algebraic models. Thus the same class of algebras can be meaningfully used to study both versions of Lewis's logics; precisely, we have strong completeness of both calculi with respect to V-algebras. We also initiate the study of the structure theory of the algebraic models; interestingly, we demonstrate that the congruences of the algebras, which are in one-one correspondence with the deductive filters inherited by the logics, can be characterized by means of the congruences of their modal reducts.

The second part of our work develops a duality result for V-algebras, circling back to the original intended sphere models. In more details, we show two different dual categorical equivalences of our algebraic structures with respect to topological spaces based respectively on Lewis's spheres and (Stalnaker's inspired) selection functions. The dualities we show are enrichments of Stone duality between Boolean algebras with homomorphisms and Stone spaces with continuous maps, where the operator $\Box\!\!\rightarrow$ is interpreted first by means of a selection function, and then by a map associating a set of nested spheres to each element of the space. The formal work developed for the dualities also allows us to demonstrate the strong completeness of sphere models with respect to Lewis's logics. Finally, thanks to the duality results, we also clarify the role of the *limit assumption*, a condition on sphere models that has been extensively discussed in the literature (see for example [6, 2]). In particular, we will see that both the strong and weak calculi are strongly complete with respect to models that do satisfy the limit assumption; in this sense, models without the limit assumption are not really "seen" by Lewis's logics.

In conclusion, this contribution is meant to provide a logico-algebraic treatment of Lewis variably strict conditional logics. Our results aim at clarifying several ambiguities in the literature surrounding these logics, explicitly defining and refining their properties and theorems, and introducing a novel general algebraic and topological framework for their technical analysis.

References

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, jun 2001.

[2] Kit Fine. Counterfactuals without possible worlds. *Journal of Philosophy*, 109(3):221–246, 2012.

[3] David Lewis. Completeness and decidability of three logics of counterfactual conditionals. *Theoria*, 37:74–85, 1971.

[4] David Lewis. *Counterfactuals*. Blackwell, 1973.

[5] Donald Nute. *Topics in Conditional Logic*. Boston, MA, USA: Reidel, 1980.

[6] John L. Pollock. *Subjunctive Reasoning*. Springer Netherlands, 1976.

[7] Giuliano Rosella, Tommaso Flaminio, and Stefano Bonzio. Counterfactuals as modal conditionals, and their probability. *Artificial Intelligence*, 323(C):103970, 2023.

[8] Robert Stalnaker. A Theory of Conditionals. In Nicholas Rescher, editor, *Studies in Logical Theory*, number 2 in American Philosophical Quarterly Monograph Series, pages 98–112. Blackwell, Oxford, 1968.

[9] Xuefeng Wen. Modal logic via global consequence. In *Logic, Language, Information, and Computation*, pages 117–132. Springer International Publishing, 2021.