The Effective Topos May be Simple Unstable

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Recent work has uncovered a fascinating connection between model theory and category theory.

- Hyland-Pitts [1]: the Turing Degrees embed effectively into the poset of Lawvere-Tierney topologies in the Effective Topos.
- Malliaris-Shelah [4]: the Turing Degrees embed effectively into Keisler's Order on simple unstable theories.

Leveraging these perspectives against each other gives two different ways of thinking about algorithmic complexity, but also gives some interesting new clues about categorical logic and model theory. By [2], we know that the Lawvere-Tierney topologies of Eff correspond to so-called "bilayered Turing Degrees". This sets up the following problem:

Problem 1. Can we modify Malliaris-Shelah's construction to embed the bilayered Turing Degrees into Keisler's Order?

Discussion 2. Why should this interest the category theorist? A key structure theorem in topos theory says: every Grothendieck topos $\mathcal{E}$ classifies a geometric theory $\mathbb{T}_{\mathcal{E}}$, and subtoposes of $\mathcal{E}$ correspond to quotients of $\mathbb{T}_{\mathcal{E}}$. It's natural to ask if this picture extends to elementary toposes and their subtoposes, but the previous structure theorem makes use of the site representation of Grothendieck toposes in a crucial way – this is not available for elementary toposes in general (e.g. the Effective Topos). A positive solution to Problem 1 means: given any LT-topology j in Eff, we can associate to it a theory $j \mapsto \mathbb{T}_j$ such that $j \leq j'$ iff $\mathbb{T}_j \trianglelefteq \mathbb{T}_{j'}$ in Keisler's Order — without recourse to the usual site representation.

Discussion 3. Why should this interest the model theorist? It is currently not known what the smallest upper bound of the Turing Degree theories are in Keisler's Order (if it even exists).[1] A positive solution to Problem 1 brings into focus a potential new dividing line within simple unstable theories: [5, Prop 3] says that if j is a non-trivial LT topology in Eff such that $j_A \leq j$ for any Turing Degree topology j_A, then $j_{\neg\neg} = j$. This would also clarify the picture of how we might understand simple theories as being built out of certain basic building blocks, raising interesting implications for viewing Keisler's Order as a systematic search for important partition patterns of set systems.

This talk will introduce and clarify the connection between these two embeddings of the Turing Degrees, with a view towards Problem 1. As a baseline step towards its solution, we focus on how both the Effective Topos & Keisler's Order use topological ideas to calibrate jumps in complexity. Time permitting, we may discuss Lee-van Oosten's result that each LT-topology in Eff is built from a family of basic LT-topologies [3], and explore its ramifications.

[1] An upper bound exists since there exists a maximal class in Keisler's Order (which includes theories satisfying SOP_1); the question is how much further down can we move this upper bound. In particular, recall that there does not exist a maximal Turing Degree.

References

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