Commutation Groups and State-Independent Contextuality

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Abstract

Contextuality is a key form of non-classicality in quantum mechanics. It has a strong logical content, and contextuality arguments are often referred to as paradoxes. They contradict the basic assumption of classical physics, that observable quantities have well-defined values independently of which measurements are performed.

The strongest form is state-independent contextuality, where the structure of the observables dictates that contextuality arises for any state. The most famous example of this phenomenon is the Peres-Mermin magic square [2], which is constructed from the 2-qubit Pauli group:

$$
\begin{array}{ccccc}
XI & — & IX & — & XX \\
| & & | & & | \\
IZ & — & ZI & — & ZZ \\
| & & | & & | \\
XZ & — & ZX & — & YY
\end{array}
$$

Here XI denotes the 2-qubit operator $\sigma_x \otimes I$, and similarly for the other entries. One can now calculate that the operators in each row and column pairwise commute, and hence form a valid measurement context. Moreover, the product of each of the rows, and of the first two columns, is II; while the product of the third column is $-II$. We can use this behaviour to show the impossibility of assigning values to the observables which respects the algebraic structure of commuting observables (*i.e.* those which can be performed together).

We now wish to abstract from the specifics of the Pauli group, and understand the general structure which makes such arguments possible. This leads us to introduce the notion of *commutation group*, to which we now turn.

The idea behind commutation groups is that they are built freely from prescribed commutation relations on a set of generators. Commutation relations play a fundamental role in quantum mechanics, the canonical example being the commutation relation between position and momentum (see e.g. [1]): $[p,q] = i\hbar\mathbb{1}$. We can think of a commutation relation as saying that two elements commute *up to a prescribed scalar*. For this to make sense in a group theoretic context, we need an action of a suitable (classical, hence abelian) group of scalars or "phases" on the group we are constructing. We are interested here in finite group constructions, so we shall work over the finite cyclic groups $\mathbb{Z}_d$, $d \geq 2$.

Given a finite set X of generators, we define a *commutator matrix* to be a map $\mu : X^2 \to \mathbb{Z}_d$ which is skew-symmetric, meaning that $\mu(x,y) = -\mu(y,x)$ for all $x, y \in X$.

We describe the construction of commutation groups from commutator matrices in two ways: by generators and relations, and by a linear algebraic construction. Both are useful, and convey different intuitions. The key relations are the commutation relations $xy \doteq J_{\mu(x,y)}yx$.

Main Results

We summarize the main results:

1. We present commutation groups by generators and relations, parameterised by a commutator matrix. We show that these groups admit a presentation by a confluent and terminating rewriting system, using a given linear order on the generators. The normal forms for this presentation lead to an isomorphism with a form of Heisenberg group [3].

2. We are interested in analyzing contextuality arguments over commutation groups. We use a notion of compatible partial monoid, which allows the idea of closing a set of generators under commuting products to be captured. The scalars embed into the centre of this generated compatible sub-monoid $\mathsf{G}(\mu)$. Non-contextual value assignments correspond to left splittings of this embedding.

3. Contextual words provide witnesses for contextuality, *i.e.* obstructions to the existence of non-contextual value assignments. They generalize the usual argument for the contextuality of Peres-Mermin and other examples in the literature.

   A contextual word is a word over the generators of the commutation group such that:

   - The word can be formed by commuting products from the generators.
   - Each generator occurs a multiple of d times in the word
   - The global phase factor of the word is non-zero.

   The existence of a contextual word implies the contextuality of the commutation group.

4. By a detailed analysis of inversions in words in $\mathsf{G}(\mu)$, we show that contextual words cannot arise in commutation groups over $\mathbb{Z}_d$ for d odd. We explicitly construct non-contextual value assignments for these cases.

5. For even d, we firstly show that if the commutativity graph of $\mathsf{G}(\mu)$ is a cluster graph, then non-contextual value assignments exist. This is shown by verifying the sheaf property for empirical models over the commutation group.

6. In the remaining cases, we give a fine-grained analysis of when contextual words exist. This relies on a reduction of commutator matrices to Darboux normal form, which can be performed even for composite d. Whenever some simple arithmetical criteria are met, contextual words can be constructed over each 4×4 block of this normal form.

7. We show that commutation groups over $\mathbb{Z}_d$ with n generators have faithful representations in the unitary group acting on n qudits. Moreover, the image of these embeddings lies inside the generalized Pauli group over n qudits. Generalized Paulis, which are isomorphic to Heisenberg groups over $\mathbb{Z}_d$, are themselves examples of commutation groups, as are groups associated with Majorana fermions.

References

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