Multi-type universal algebra: Transfer of properties *

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Multi-type algebras are special type of *heterogeneous algebras* used in previous research to provide multi-type algebraic semantics for logics such as Semi De Morgan logic [2], and linear logic [3]. This class of algebras dually corresponds to the multi-type logic equivalent to the single-type logic under consideration. These studies were conducted to overcome some problematic issues related to the proof-theoretic perspective on those given logic systems which are not properly displayable in the sense of Wansing [5]. The present paper aims to capture the general construction of multi-type algebras and to study their universal algebraic properties.

Definition 1. *A multi-type algebra $\mathbb{H}$ is a tuple $(\mathbb{A}, \mathbb{K}, h, e)$ such that*
1. $\mathbb{A}$ and $\mathbb{K}$ are some (single-type) algebras.
2. $h : \mathbb{A} \to \mathbb{K}$ is a surjective map and $e : \mathbb{K} \to \mathbb{A}$ is an injective map.
3. The set of operations $\{f^{\mathbb{K}} \mid f \in \mathcal{O}_{\mathbb{A}}\}$ is a subset of the basic operations $\mathcal{O}_{\mathbb{K}}$ on the kernel $\mathbb{K}$, where for any n-ary operation $f \in \mathcal{O}_{\mathbb{A}}$, and $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$,

$$f^{\mathbb{K}}(\alpha_1, \alpha_2, \cdots, \alpha_n) = hf(e(\alpha_1), e(\alpha_2), \cdots, e(\alpha_n)).$$

Given any multi-type algebras $\mathbb{H}_1 = (\mathbb{A}_1, \mathbb{K}_1, h_1, e_1)$ and $\mathbb{H}_2 = (\mathbb{A}_2, \mathbb{K}_2, h_2, e_2)$, a multi-type homomorphism between $\mathbb{H}_1$ and $\mathbb{H}_2$ is a pair $F = (F^{\mathbb{A}}, F^{\mathbb{K}})$, such that $F^{\mathbb{A}} : \mathbb{A}_1 \to \mathbb{A}_2$ and $F^{\mathbb{K}} : \mathbb{K}_1 \to \mathbb{K}_2$ are homomorphisms of these algebras as single-type algebras, and $F^{\mathbb{K}} \circ h_1 = h_2 \circ F^{\mathbb{A}}$ and $F^{\mathbb{A}} \circ e_1 = e_2 \circ F^{\mathbb{K}}$. We define multi-type representation of a single-type algebra as follows.

Definition 2. *Let $\mathcal{K}$ be a class of algebra with set of the basic operations $\mathcal{O}$ on the algebras in $\mathcal{K}$. Let $\sigma \in \mathcal{O}$ be a unary operation on the algebras in $\mathcal{K}$. Suppose there exist disjoint set of the basic operations $\mathcal{O}_1$, $\mathcal{O}_2$, and $\{\sigma\}$ such that $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \{\sigma\} = \mathcal{O}$. Then the multi-type representation of any $\mathbb{A} \in \mathcal{K}$, $Mult_{(\sigma,\mathcal{O}_1,\mathcal{O}_2)}(\mathbb{A}) = (\mathbb{A}', \mathbb{K}, h, e)$, where*
1. $\mathbb{A}'$ is an $\mathcal{O}_1$-reduct of $\mathbb{A}$,
2. $\mathbb{K} = (\sigma(A), \{f^{\mathbb{K}} \mid f \in \mathcal{O}\})$,
3. $h : \mathbb{A}' \to \mathbb{K}$ and $e : \mathbb{K} \to \mathbb{A}'$ are surjective and injective maps respectively, such that for any $\alpha \in \sigma(A)$, $e(\alpha) = \alpha$ and $e \circ h = \sigma$.

In Definition 2, the algebra $\mathbb{K}$ is called the kernel of the multi-type representation of $\mathbb{A}$. We also define the multi-type representation of a homomorphism between two single-type algebras $F : \mathbb{A}_1 \to \mathbb{A}_2$ as $Mult_{(\sigma,\mathcal{O}_1,\mathcal{O}_2)}(F) = (F^{\mathbb{A}}, F^{\mathbb{K}})$ where $F^{\mathbb{K}} : \mathbb{K}_1 \to \mathbb{K}_2$ and $F^{\mathbb{A}} : \mathbb{A}_1 \to \mathbb{A}_2$ such that for any $\alpha \in \mathbb{K}_1$, and $a \in \mathbb{A}_1$, $F^{\mathbb{K}}(\alpha) = h_2 F e_1(\alpha)$ and $F^{\mathbb{A}}(a) = F(a)$. For a given class of single-type algebras acting as algebraic semantics, this construction allows us to define a class of multi-type algebras that serve as multi-type algebraic semantics for the same logic. It can be shown that any category of single-type algebras is categorically equivalent to the category of multi-type algebras defined from it by the above construction.

The multi-type construction transfers many universal algebraic properties of the single-type algebra $\mathbb{A}$ to the kernel of its multi-type representation under certain conditions. In this work, we investigate this phenomenon and show several properties which are transferred in this manner. Table 1 depicts some properties of the kernels and corresponding conditions on the single-type

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	Property in $\mathbb{K}$	**Conditions on $\mathbb{A}$**	**Side conditions**
1	Distributivity	$\sigma(\sigma a \wedge \sigma(\sigma b \vee \sigma c)) = \sigma(\sigma(\sigma a \wedge \sigma b) \vee \sigma(\sigma a \wedge \sigma c))$	-
2	$f^{\mathbb{K}}(\alpha \sqcup^{\epsilon} \beta) = f^{\mathbb{K}}(\alpha) \sqcup f^{\mathbb{K}}(\beta)$	$f(\sigma a \vee^{\epsilon} \sigma(b)) = \sigma(f(a) \vee f(b))$	$\sigma f(\sigma a) = f(a)$
3	$f^{\mathbb{K}}(\alpha \sqcap^{\epsilon} \beta) = f^{\mathbb{K}}(\alpha) \sqcap f^{\mathbb{K}}(\beta)$	$f(\sigma a \wedge^{\epsilon} \sigma(b)) = \sigma(f(a) \wedge f(b))$	$\sigma f(\sigma a) = f(a)$
4	Injective (surjective) homomorphism	Injective (surjective) homomorphism	-
5	Congruence permutable	congruence permutable	-
6	Congruence distributive	congruence distributive	-
7	Arithmetical	arithmetical	-
8	Amalgamation property	amalgamation property	$H(\mathcal{V}') \subseteq Ker(H(\mathcal{V}))$
9	Superamalgamation property	superamalgamation property	$H(\mathcal{V}') \subseteq Ker(H(\mathcal{V}))$

Table 1: Properties of kernels and corresponding condition on single type algebras. The $\epsilon \in \{\partial, 1\}$ where $\vee^{\epsilon} = \vee$ and $\sqcup^{\epsilon} = \sqcup$ when $\epsilon = 1$, and $\vee^{\epsilon} = \wedge$ and $\sqcup^{\epsilon} = \sqcap$ when $\epsilon = \partial$. Similarly, $\wedge^{\epsilon} = \wedge$ and $\sqcap^{\epsilon} = \sqcap$ when $\epsilon = 1$, and $\wedge^{\epsilon} = \vee$ and $\sqcap^{\epsilon} = \sqcup$ when $\epsilon = \partial$.

algebras along with some side conditions on operator σ which imply the given property for the kernel. We have a particular interest in algebras with (semi) lattice structure with monotone and idempotent unary operator σ, as these algebras often provide algebraic semantics for commonly studied logics. Under this assumption, we define an order $\leq_{\mathbb{K}}$ on the kernel as follows: for any $\alpha, \beta \in \mathbb{K}$, $\alpha \leq_{\mathbb{K}} \beta$ iff $e(\alpha) \leq e(\beta)$ where $\leq$ is the standard order on $\mathbb{A}$. Under the order $\leq_{\mathbb{K}}$, the algebra $\mathbb{K}$ forms a lattice with join and meet defined as $\alpha \sqcup \beta := h(e(\alpha) \vee e(\beta))$ and $\alpha \sqcap \beta := h(e(\alpha) \wedge e(\beta))$, respectively. The first three items in Table 1 relate order theoretic-properties of single-type algebras with lattice structure to those of their kernels. The fourth item relates properties of single-type and kernel homomorphisms. The remaining items pertain to the case when the class of single-type algebras $\mathcal{V}$ and the class of their kernels $\mathcal{V}'$ form varieties. For items 5 to 7 by Malcev Conditions [1], these properties are equivalent to the existence of corresponding Malcev terms. These items are proven by showing that the existence of Malcev terms on $\mathcal{V}$ implies the existence of those terms on $\mathcal{V}'$. For items 8 and 9, $H(\mathcal{V})$ and $H(\mathcal{V}')$ denote the sets of all homomorphisms on $\mathcal{V}$ and $\mathcal{V}'$, respectively, and the map Ker assigns a $\mathcal{V}$-homomorphism to the kernel component of its multi-type representation. Amalgamation and superamalgamation properties are important because they dually correspond to certain interpolation properties [4] of the logic of the given variety.

Moreover, any congruence relation θ on $\mathbb{A}$ defines a congruence relation $\theta_{\mathbb{K}}$ on $\mathbb{K}$, defined by $\alpha \theta_{\mathbb{K}} \beta$ iff $e(\alpha)\theta e(\beta)$. This defines a lattice homomorphism κ between $Con(\mathbb{A})$ and $Con(\mathbb{K})$ given by $\kappa : Con(\mathbb{A}) \to Con(\mathbb{K})$ which assigns any $\theta \in Con(\mathbb{A})$ to $\kappa(\theta) = \theta_{\mathbb{K}}$. This relationship between congruence lattices is important in studying conditions under which the class of kernels defined by a variety of single-type algebras, forms a variety of multi-type algebras.

For future direction, we would like to investigate whether the results mentioned above can be derived from the properties of functor $Ker_{(\sigma, \mathcal{O}_1, \mathcal{O}_2)}$, which assigns a single-type algebra to kernel of its multi-type representation and a single-type homomorphism to the kernel component of its multi-type representation.

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