Amalgamation in Varieties of BL-algebras

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BL-algebras are the equivalent algebraic semantics of Hájek's basic fuzzy logic **BL**. The latter was shown in [7] to be the logic of continuous t-norms, and **BL** and BL-algebras have subsequently attracted a great deal of attention from both fuzzy and substructural logicians. In the last twenty-five years, an extensive body of work on **BL** and BL-algebras has developed and the literature now offers a rather mature theory (see, e.g., [1, 2, 5, 6, 9] for a sample). One outstanding problem in this theory, however, is to completely classify the varieties of BL-algebras that have the amalgamation property—or, in an equivalent logical formulation, to completely classify the axiomatic extensions of **BL** that have the deductive interpolation property.

This problem has proven quite challenging. In [12], Montagna showed that many of the most natural varieties of BL-algebras (including the variety of all BL-algebras) have the amalgamation property, but that there are uncountably many varieties of BL-algebras that do not. Later, by working with some technical hypotheses on the form of generating algebras for the varieties in question, Aguzzoli and Bianchi provided a partial classification of varieties of BL-algebras with the amalgamation property [3]. They later sharpened this classification in [4], but their results still stopped short of an exhaustive classification.

In this work, we provide just such an exhaustive classification of varieties of BL-algebras that have the amalgamation property, consequently giving a complete classification of the axiomatic extensions of Hájek's basic logic that have the deductive interpolation property. In particular, we show that there are just countably many of these, answering the question posed by Montagna in [12, Section 7].

Our classification proceeds in three steps. First, we obtain a new equivalent formulation of the amalgamation property that is better suited to studying amalgamation in many varieties generated by linearly ordered algebras, including varieties of BL-algebras. We say that an extension $\mathbf{A} \leq \mathbf{B}$ is *essential* provided that $\theta \neq \Delta_B$ implies $\theta \cap A^2 \neq \Delta_A$, and we say that an embedding $\phi\colon \mathbf{A} \to \mathbf{B}$ is *essential* whenever $\phi[\mathbf{A}] \leq \mathbf{B}$ is. A span $\langle i_1\colon \mathbf{A} \to \mathbf{B}, i_2\colon \mathbf{A} \to \mathbf{C}\rangle$ of algebras is *essential* provided that i_2 is an essential embedding, and a class of algebras K has the *essential amalgamation property* if for any essential span $\langle i_1\colon \mathbf{A} \to \mathbf{B}, i_2\colon \mathbf{A} \to \mathbf{C}\rangle$ in K, there exists $\mathbf{D} \in \mathsf{K}$ and embeddings $j_1\colon \mathbf{B} \to \mathbf{D}$ and $j_2\colon \mathbf{C} \to \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$.

Theorem 1. *Let V be a variety and $\mathsf{V}_{\mathrm{FSI}}$ be the class of finitely subdirectly irreducible members of V. Suppose that V has the congruence extension property and that $\mathsf{V}_{\mathrm{FSI}}$ is closed under subalgebras and homomorphic images. Then V has the amalgamation property if and only if $\mathsf{V}_{\mathrm{FSI}}$ has the essential amalgamation property.*

In the second step toward our classification, we apply the previous theorem to study amalgamation for 0-free subreducts of BL-algebras, often called *basic hoops*. The finitely subdirectly irreducible basic hoops are precisely the totally ordered ones, so Theorem 1 provides a powerful criterion for the amalgamation property in this context. Together with the well-known decomposition of totally ordered BL-algebras as ordinal sums of Wajsberg hoops and the classification of varieties of Wajsberg hoops with the amalgamation property [11, Theorem 63], we use Theorem 1 to give a tangible description of the poset of varieties of basic hoops with the amalgamation property. In particular, we show that this poset can be partitioned into a countably infinite family of finite intervals and give concrete descriptions of the latter. Thus:

Theorem 2. *There are only countably many varieties of basic hoops that have the amalgamation property.*

In the third step toward our classification, we use Theorem 2 along with the well-known classification of varieties of MV-algebras with the amalgamation property (see [8]) to describe all varieties of BL-algebras with the amalgamation property. Like for basic hoops, these turn out to all fall into one of countably infinitely many finite intervals, which we may concretely describe. Thus:

Theorem 3. *There are only countably many varieties of BL-algebras that have the amalgamation property.*

By applying the well-known connection between the amalgamation property and the deductive interpolation property for algebraizable logics, we may deduce the following result from Theorems 2 and 3.

Theorem 4. *There are only countably many axiomatic extensions of Hájek's basic fuzzy logic that have the deductive interpolation property. The same holds for axiomatic extensions of the negation-free fragment of Hájek's basic fuzzy logic.*

More information can be found in our preprint [10].

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