The Kuratowski's Problem in Pointfree Topology

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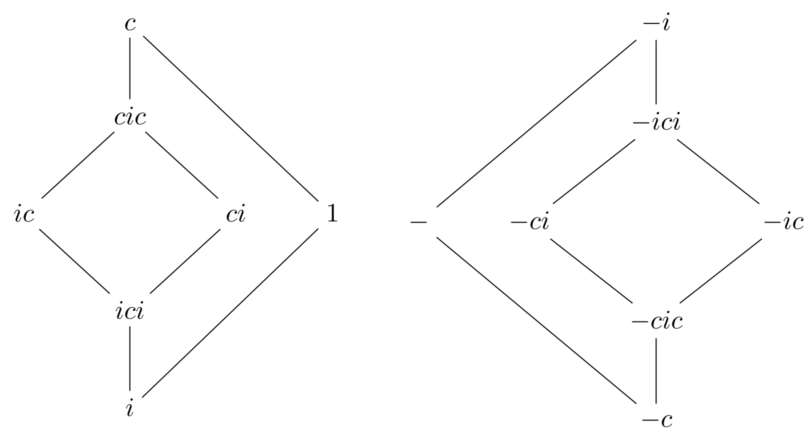
A classic result of Kuratowski states that there are at most 7 distinct combinations of the operators of interior (i) and closure (c) on a topological space, which become 14 if also the set-theoretic complement ($-$) is considered.[1] These operators form an ordered monoid w.r.t. composition and pointwise ordering, the so-called Kuratowski's monoid, whose Hasse diagram is shown below:



(where 1 is the identity operator). Special classes of spaces can be characterized by the fact that two or more of these operators coincide [4]; for instance, a space whose open sets form a complete Boolean algebra satisfies the equation $ici = i$.

What happens to this picture if it is looked at from a constructive point of view?
And what about the pointfree (i.e. localic) version of the Kuratowski's problem?

We answer both of these questions and we explain why they are related to each other.

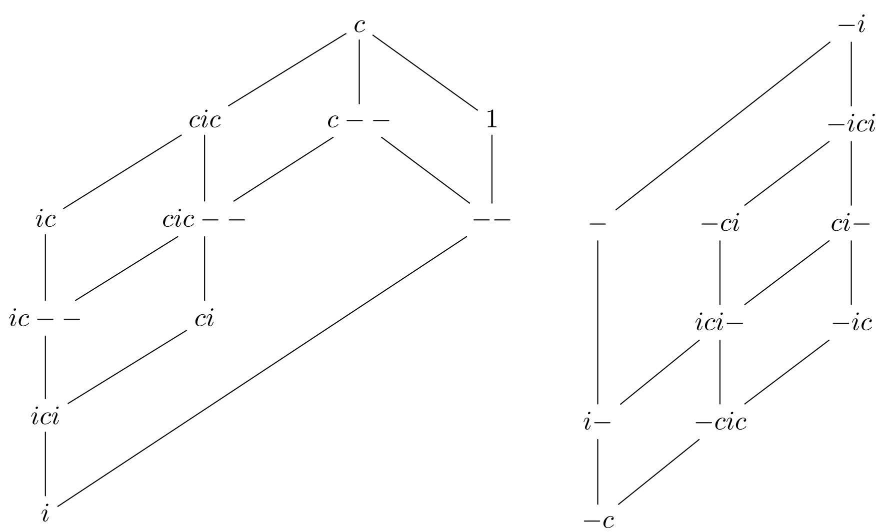
First, we recall a constructive account of the closure-interior problem (that is, the one not involving the set-theoretic complement) that we know from Giovanni Sambin [3]. For the sake of generality, we consider a closure operator and an interior operator on an arbitrary poset. It turns out that the ordered monoid generated by i and c in such a framework depends neither on the Law of Excluded Middle nor on topological notions as strictly understood.

In a constructive setting, the collection of subsets of a given set is only a frame (a.k.a. a complete Heyting algebra), instead of a complete Boolean algebra, and the set-theoretic "complement" is only a pseudocomplement. This naturally poses a general version of the Kuratowski's problem on an arbitrary frame, which has potential applications in constructive modal

[1] In logical terms, this means that there are exactly 13 different modalities (not counting identity) in $S4$.

logic. However, the presence of the pseudocomplement greatly increases the number of possible combinations [1]. To simplify the matter we restrict to the case in which $c = -i-$ (this equation is constructively true in all topological spaces, although its dual $i = -c-$ is not): this we call the interior-pseudocomplement problem on a frame. Contrary to the Boolean case, we get 31 possible combinations (instead of 14) that apparently are all different [2], in general.

This constructive result can be applied to solve the Kuratowski's problem in a pointfree framework, that is, within the theory of locales. Indeed, it is well known that the sublocales of a given locale form a co-frame (the opposite of the frame of nuclei); in particular, every sublocale has a co-pseudocomplement. Moreover, the usual notion of an open (closed) sublocale gives rise to an interior (a closure) operator on the co-frame of sublocales.[2] So the Kuratowki's problem for sublocales is related, although in a dual way, to the constructive interior-pseudocomplement problem discussed above. We can therefore apply the previous result and, thanks to some specific properties of open/closed sublocales, we can lower the number of possible combinations of interior, closure and co-pseudocomplement to 21 (see the picture below).



Showing that this picture cannot be further simplified is still an open problem [2].

References

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[3] F. Ciraulo and G. Sambin. The overlap algebra of regular opens. *J. Pure Appl. Algebra*, 214(11):1988–1995, 2010.

[4] B. J. Gardner and M. Jackson. The Kuratowski closure-complement theorem. *New Zealand J. Math.*, 38:9–44, 2008.

[2] The interior (closure) operator on sublocales corresponds to a closure (interior) operator on nuclei.