The maximal spectrum of d-elements is not always Hausdorff

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The d-ideals play an important role in the study of Riesz spaces (see, e.g., [3]). They are exactly the fixpoints of a nucleus on the arithmetic frame of all ideals of a Riesz space. Martinez and Zenk [4] initiated a general study of this nucleus on an arbitrary arithmetic frame. They coined it as the *d-nucleus*. The d-nucleus and its corresponding sublocale were further studied by Bhattacharjee [2], who initiated the study of the spectrum of maximal d-elements. This spectrum is always a locally compact T_1-space, but the question of whether it is Hausdorff was left open.

The aim of this talk is to solve this question in the negative, as well as to give a characterization of when the spectrum is Hausdorff. Our main tool is Priestley duality for the category of bounded distributive lattices [5, 6], and especially its restriction to the category of frames [7, 8]. More specifically, we will utilize Priestley duality for arithmetic frames described in [1].

Let L be an arithmetic frame. For $a \in L$, we write a^* for the pseudocomplement of a in L and define the d-nucleus $d : L \to L$ by

$$da = \bigvee \{k^{**} \mid k \text{ is compact and } k \leq a\}.$$

Let L_d be the sublocale of L of the d-fixpoints. We write X for the Priestley space of L and X_d for the Priestley space of L_d. (Note that $X_d \subseteq X$.)

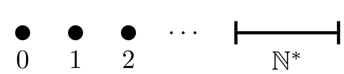
Let Y be the *localic part* of X (the space of points of L). The localic part of X_d is given by $Y_d = X_d \cap Y$. Since $\mathrm{cl}(Y_d) = X_d$, it is especially important to understand the localic part of X_d. It turns out that $y \in Y_d$ iff y is a relatively maximal localic point of X in the following sense:

Lemma 1. *$y \in Y_d$ iff y is the greatest localic point below a maximal point of X.*

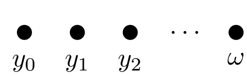
Let $\max(L_d)$ be the spectrum of maximal d-elements [2]. The above lemma gives us means to identify $\max(L_d)$ inside X. of Y. In fact, it is the set Let $\min(Y_d)$ be the set of minimal localic points of X_d.

Theorem 2. *$\max(L_d)$ is homeomorphic to $\min(Y_d)$.*

We produce an example of the Priestley space X of an arithmetic frame L such that $\min(Y_d)$ is not Hausdorff. The strategy is to construct a space where $\min(Y_d)$ is homeomorphic to the natural numbers with the cofinite topology. We achieve this as follows. Take the disjoint union of the Stone-Cêch compactification

$\beta\mathbb{N} =$ 

and the one-point compactification



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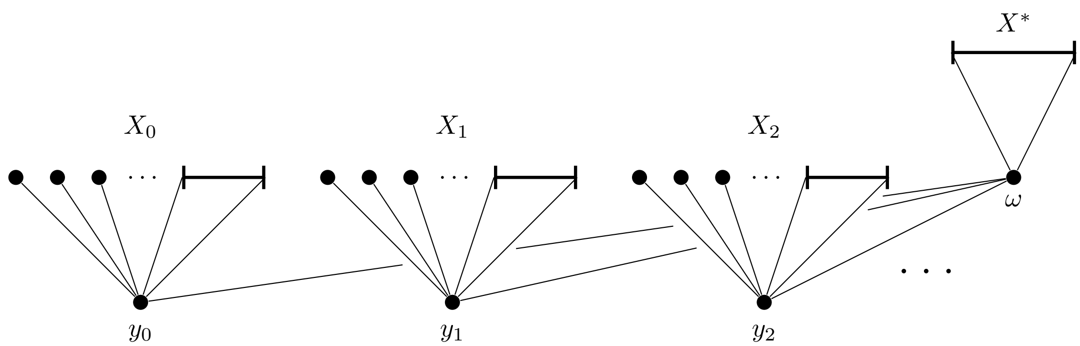
of the natural numbers. Then partition $\beta\mathbb{N} = (\bigcup X_i) \cup X^*$ into infinitely many copies X_i of $\beta\mathbb{N}$ and a subset $X^* \subseteq \mathbb{N}^*$. Equipped with the order in the diagram below, we obtain the Priestley space of an arithmetic frame such that $\min(Y_d) = \{y_0, y_1, \dots\}$ is the desired non-Hausdorff space.



Corollary 3. *There are arithmetic frames L such that* $\max(L_d)$ *is not Hausdorff.*

It is worth pointing out that $\max(L_d)$ in the above example is not even sober (recall that a topological space is *sober* if each irreducible closed set is the closure of a unique point). In general, sobriety is strictly weaker than Hausdorffness (i.e., every Hausdorff space is sober, but not vice versa). However, in the case of $\min(Y_d)$, sobriety and Hausdorffness become equivalent properties, thus yielding our characterization:

Theorem 4. $\min(Y_d)$ *is Hausdorff iff* $\min(Y_d)$ *is sober.*

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