McKinsey-Tarski Algebras

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In this talk we report on our findings in [3, 4], where an alternate pointfree approach to topology was developed, based on the work of McKinsey and Tarski [10]. We introduce the category **MT** of McKinsey-Tarski algebras and show that it provides a faithful generalization of both **Top** (the category of topological spaces) and **Frm** (the category of frames).

Definition 1.

(1) A *McKinsey-Tarski algebra* (or *MT-algebra* for short) is a pair $M = (B, \Box)$, where B is a complete Boolean algebra and $\Box$ is an interior operator on B (that is, $\Box$ satisfies the Kuratowski axioms $\Box 1 = 1$, $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box a \leq a$, and $\Box a \leq \Box\Box a$).

(2) An *MT-morphism* between MT-algebras M and N is a complete Boolean homomorphism $h : M \to N$ such that $h(\Box_M a) \leq \Box_N h(a)$ for each $a \in M$.

(3) Let **MT** be the category of MT-algebras and MT-morphisms.

Remark 2.

(1) The study of interior algebras was initiated by McKinsey and Tarski [10]. Interior algebras play an important role in modal logic as they are algebraic models of the well-known modal system **S4** (see, e.g., [11, 5]). MT-algebras are nothing more but complete interior algebras.

(2) MT-morphisms are not homomorphisms of interior algebras, but it is the inequality condition in the above definition that provides a faithful generalization of continuous maps (see [2, 6]). Such morphisms are known as stable homomorphisms (see [1]).

Connection between MT and Frm: Let $M \in$ **MT**. Call an element $a \in M$ *open* if $a = \Box a$. Let $\mathrm{O}(M)$ be the collection of open elements of M. Then $\mathrm{O}(M) \in$ **Frm** and this correspondence extends to a functor O : **MT** $\to$ **Frm**. It is a consequence of Funayama's theorem that O : **MT** $\to$ **Frm** is essentially surjective. However, this does **not** give rise to a functor from **Frm** to **MT**.

Connection to Top: Canonical examples of MT-algebras come from topological spaces. For each $X \in$ **Top**, we have that $(\mathcal{P}(X), \mathrm{int}) \in$ **MT**, and this correspondence gives rise to a contravariant functor $\mathcal{P}$: **Top** $\to$ **MT**. Its contravariant adjoint is given by the functor at : **MT** $\to$ **Top** which maps each MT-algebra M to the space $at(M)$ of atoms equipped with the topology $\eta[\mathrm{O}(M)]$, where $\eta(a) = \{x \in at(M) \mid x \leq a\}$. This gives rise to the contravariant adjunction $(\mathcal{P}, at)$, which restricts to a dual equivalence between **Top** and the reflective subcategory of **MT** consisting of atomic MT-algebras.

Separation axioms in MT-algebras: We generalize the well-known separation axioms for topological spaces and frames to MT-algebras by describing them in terms of the embedding $\mathrm{O}(M) \hookrightarrow M$.

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Sobriety and local compactness: We derive an analogue of the Hofmann-Mislove theorem [8] for sober MT-algebras. Utilizing this result, we establish the MT counterparts of Hofmann-Lawson duality [7] between locally compact frames and locally compact sober spaces and Isbell duality [9] between compact regular frames and compact Hausdorff spaces.

Stone duality: The celebrated Stone duality establishes that the category **BA** of boolean algebras is dually equivalent to the category **Stone** of Stone spaces. We define the category **StoneMT** of Stone MT-algebras and show that it is equivalent to both **BA** and the category **StoneFrm** of Stone frames. The equivalence between **StoneFrm** and **StoneMT** is obtained by restricting O. The equivalence between **StoneMT** and **BA** is established as follows.

The functor Clp : **StoneMT** $\to$ **BA** associates with each MT-algebra M the boolean algebra of clopen elements of M. A quasi-inverse of Clp : **StoneMT** $\to$ **BA** is the functor $(-)^\sigma : \mathbf{BA} \to \mathbf{StoneMT}$ which associates with each boolean algebra B the Stone MT-algebra $M = (B^\sigma, \Box)$, where B^σ is the canonical extension of B and $\Box : B^\sigma \to B^\sigma$ is defined by $\Box x = \bigvee\{b \in B \mid b \le x\}$.

$$\mathbf{StoneFrm} \xleftarrow{\quad \mathrm{O} \quad} \mathbf{StoneMT} \underset{\mathrm{Clp}}{\overset{(-)^\sigma}{\rightleftarrows}} \mathbf{BA}$$

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