On abstract model theory and logical topologies*

Asterios Gkantzounis[1], Foivos Skarpelos[1], Yiannis Kiouvrekis[2], and Petros Stefaneas[1]

[1] Department of Mathematics, School of Applied Mathematical and Physical Sciences, NTUA, Athens
minasteris@gmail.com, f-skarp@hotmail.com, petros@math.ntua.gr

[2] Mathematics, Computer Science and Artificial Intelligence Lab, Faculty of Public and One Health, University of Thessaly, Greece
kiouvrekis.y@uth.gr, yiannis.kiouvrekis@gmail.com

Abstract

The intersection of mathematical logic and topology has been a vibrant area of research for several decades. McKinsey and Tarski's seminal work on the topological interpretation of modal logic [11, 12], played a crucial role in establishing the relationship between logic and topology. Additionally, McKinsey and Tarski developed an algebraic and topological framework for Intuitionistic Logic (IL) and Modal Logics (ML) [12, 13], demonstrating that topological spaces can serve as interpretation models for IL and ML. Tarski further proved that S4 is complete with respect to topological spaces [16], while McKinsey and Tarski showed that S4 is the modal logic of real numbers in 1944 [12]. Various articles offer additional insights into these topics [3, 2, 14], and alternative approaches can be found in the works of Lawvere [8] and Goldblatt [5]. In Universal Logic, Lewitzka presents a different approach to logical systems [9, 10], constructing a theory of logical representations (a logic map) to leverage the fact that every logical system can define a topology within its theory set. In this abstract we propose the study of logic and topology independently from the underlying logic.

Our approach is based on the theory of institutions. Institutions constitute the main branch of the categorical abstract model theory, which formalizes the notion of a logical system, including syntax, semantics and the satisfaction relation between them [1]. An *institution* $\mathcal{I} = \left(\mathbb{S}ig^{\mathcal{I}}, \boldsymbol{Sen}^{\mathcal{I}}, \boldsymbol{Mod}^{\mathcal{I}}, \models^{\mathcal{I}}\right)$ consists of (a) a category $\mathbb{S}ig^{\mathcal{I}}$, the objects of which are called *signatures*; (b) a functor $\boldsymbol{Sen}^{\mathcal{I}} : \mathbb{S}ig^{\mathcal{I}} \to \mathbb{S}et$ such that it assigns a set the elements of which are called *sentences* over each signature; (c) a functor $\mathbf{Mod}^{\mathcal{I}} : \left(\mathbb{S}ig^{\mathcal{I}}\right)^{op} \to \mathbb{CAT}$ giving a category the objects of which are called Σ-*models* and the arrows of which are called Σ-*morphisms* for each signature Σ, and (d) a relation $\models^{\mathcal{I}}_{\Sigma} \subseteq \left|Mod^{\mathcal{I}}(\Sigma)\right| \times \boldsymbol{Sen}^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in \left|\mathbb{S}ig^{\mathcal{I}}\right|$, called Σ-*satisfaction* such that for each morphism $\phi : \Sigma \to \Sigma'$ in $\mathbb{S}ig^{\mathcal{I}}$, the *satisfaction condition*.

For every signature Σ we define a class of topologies over the category of models $\mathbf{Mod}(\Sigma)$ based on the class of subsets of $\boldsymbol{Sen}(\Sigma)$, this class being closed under union. We consider the morphisms of model categories $\mathbf{Mod}(\phi) : \mathbf{Mod}(\Sigma') \to \mathbf{Mod}(\Sigma)$ induced by signature morphisms $\phi : \Sigma' \to \Sigma$ standing for change of notation. We investigate the broader possible class of topologies in $\mathbf{Mod}(\Sigma)$ whose members are mapped to topologies in $\mathbf{Mod}(\Sigma')$ via the arrow $\mathbf{Mod}^{-1}(\phi)$, as well as the broader possible class of topologies in $\mathbf{Mod}(\Sigma')$ whose members are mapped to topologies in $\mathbf{Mod}(\Sigma)$ via $\mathbf{Mod}(\phi)$ [7]. Furthermore, we investigate under which circumstances (ie additional properties of such morphisms) an additional structure of topologies is preserved. The questions that arise from this inquiry are on the model theoretic properties of these topologies. We prove several theorems in this direction, such as that the class of topologies includes topologies defined over categories of elementary equivalent models (ie. models that satisfy the same

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sentences), and the essential link between elementary equivalent models and the intersection preserving properties. Finally, we attempt to generalize our inquiry to Grothendieck topologies, given institutions with the appropriate categorical properties.

These findings contribute and generalize the results from [6, 15] and new results from [4]. In [6], the author has introduced the notion of topological semantics in the framework of abstract model theory through institution-independent theory. Within this framework, semantic completeness can be explored through topological concepts.

References

[1] Răzvan Diaconescu. Institution-independent Model Theory. Springer, New York, 2008.

[2] Christopher J. Eagle, Clovis Hamel, and Franklin D. Tall. Two applications of topology to model theory. Annals of Pure and Applied Logic, 172(5):102907, 2021. 50 Years of Set theory in Toronto.

[3] Leo Esakia. Diagonal constructions löb's formula and cantor's scattered space. Logical and Semantical Studies, pages 128–143, 1981.

[4] Asterios Gkantzounis. Algebraic and geometric methods for the modelling of systems. In PhD in progress, Athens, NTUA, Department of Mathematics.

[5] Robert I. Goldblatt. Grothendieck topology as geometric modality. Mathematical Logic Quarterly, 27(31-35):495–529, 1981.

[6] Yiannis Kiouvrekis. Topological models, computer science semantics and applications of logic in ai. PhD Thesis, Athens, NTUA, Department of Mathematics, 2019.

[7] Yiannis Kiouvrekis and Petros Stefaneas. Topological semantics in institutions with proofs. In A.G.Pinus, K.N.Ponomaryov, S.V.Sudoplatov, and E.I.Timoshenko, editors, Algebra and Model Theory 10, pages 92–100, Novosibirsk, 2015. NSTU.

[8] F. Lawvere. Quantifiers and sheaves. Actes du Congre International des Mathematiciens, pages 329–334, 1970.

[9] Steffen Lewitzka. Abstract logics, logic maps, and logic homomorphisms. Logica Universalis, 1(2):243–276, 2007.

[10] Steffen Lewitzka. A topological approach to universal logic: Model-theoretical abstract logics. Logica Universalis, pages 35–61, 2007.

[11] J. C. C. McKinsey. A solution of the decision problem for the lewis systems s2 and s4, with an application to topology. The Journal of Symbolic Logic, 6(4):117–134, 1941.

[12] J. C. C. McKinsey and Alfred Tarski. The algebra of topology. Annals of Mathematics, 45(1):141–191, 1944.

[13] J. C. C. McKinsey and Alfred Tarski. On closed elements in closure algebras. Annals of Mathematics, 47(1):122–162, 1946.

[14] Valentin Shehtman. Modal logics of domains on the real plane. Studia Logica, 42:63–80, 03 1983.

[15] Sergey V. Sudoplatov, Yiannis Kiouvrekis, and Petros Stefaneas. Generic constructions and generic limits. Algebraic Modeling of Topological and Computational Structures and Applications, pages 375–398, 2017.

[16] Alfred Tarski. Der aussagenkalkul und die topologie. Fund. Math., 31:103–134, 1938.