# A mixed logic with binary operators*

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## 1 Introduction

Betweenness relations—well-known from geometry—are probably the most deeply studied ternary relations in logic and mathematics. As the class of betweenness relations is not modally definable, to investigate algebraic properties of betweenness, in (Düntsch et al., 2023) we worked with the so-called *PS-algebras*, i.e., expansions of the standard (binary) modal possibility algebras $\langle A, f\rangle$ with a binary *sufficiency* operator $g\colon A \times A \to A$ satisfying the following two conditions:

(i) if $x = \mathbf{0}$ or $y = \mathbf{0}$, then $g(x, y) = \mathbf{1}$ (co-normality),

(ii) $g(x, y) \cdot g(x, z) = g(x, y + z)$ and $g(y, x) \cdot g(z, x) = g(y + z, x)$ (co-additivity).

From the outset, there is no connection between the operators $f$ and $g$. Thus, any meaningful interplay between them must be forced explicitly. One of the possible solutions is expanding the axioms with the following first-order condition

$$a \neq \mathbf{0} \text{ and } b \neq \mathbf{0} \rightarrow g(a, b) \leq f(a, b). \tag{wMIA}$$

PS-algebras satisfying (wMIA) are called *weak mixed algebras* or just *weak MIAs*. In (Düntsch et al., 2023), we showed that binary weak MIAs—counterparts of the unary wMIAs introduced by Düntsch et al. (2017)—can algebraically express certain axioms of betweenness relations.

## 2 The binary logic $K^{\#}$

The algebras from (Düntsch et al., 2017) have the following interesting property: the elements of the equational class generated by the weak MIAs are the algebraic models of the logic $K^{\sim}$, presented by Gargov et al. (1987). Our first objective is to develop the logic $K^{\#}$ (the counterpart of $K^{\sim}$) for the binary case using the copying construction of Vakarelov (1989) adapted for our needs. This is a Boolean logic with a set $\mathsf{Var}$ of propositional variables, a constant $\top$, and two extra binary modalities $\square$ and $\boxbar$ with duals $\Diamond$ and $\diamondbar$. A *ternary frame* is a structure $\mathfrak{F} \coloneqq \langle W, R, S\rangle$ where $W$ is a nonempty set, and $R, S$ are ternary relations on $W$. $\mathfrak{F}$ is called a *weak MIA frame*, if $S \subseteq R$. In (Düntsch et al., 2023), it was proved that the complex algebra of a weak MIA frame is a weak MIA and that the canonical frame of a weak MIA is a weak MIA frame. The class of weak MIA frames is decisive in the determination of the relational models of the logic $K^{\#}$. Indeed, we will prove the following theorems:

**Theorem 2.1.** *$K^{\#}$ is sound and complete with respect to wMIA frames.*

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Models are structures $\mathfrak{M} \coloneqq \langle W, R, S, v\rangle$ where $\langle W, R, S\rangle$ is a weak MIA frame and $v\colon \mathsf{Var} \to 2^W$ is a valuation. A model $\mathfrak{M}$ is called *special*, if $R = S$.

**Theorem 2.2.** *If $\mathfrak{M} \coloneqq \langle W, R, S, v\rangle$ is a model of $K^{\#}$, then there is a special model $\underline{\mathfrak{M}} \coloneqq \langle \underline{W}, \underline{R}, \underline{v}\rangle$ such that $\mathfrak{M}$ and $\underline{\mathfrak{M}}$ are modally equivalent.*

# 3 The class Eq(wMIA)

Let **wMIA** be the class of (binary) weak MIAs. Our second objective is to exhibit an axiom system for the equational class **V** of algebraic models of $K^{\#}$ and to prove that $\mathbf{V} = \mathbf{Eq}(\mathbf{wMIA})$, i.e., **V** is the variety generated by **wMIA**.

In the case of unary modalities investigated in (Düntsch et al., 2017) a unary PS-algebra $\langle A, f, g\rangle$ is a weak MIA if and only if the mapping defined by $u'(a) \coloneqq f^{\partial}(a) \cdot g(-a)$ is the dual of the unary discriminator. We have shown in (Düntsch et al., 2023) that for binary modalities such equivalence does not hold any more, and the weaker condition (**di**)

$$(\forall a, b \in A)[a \cdot b \neq \mathbf{0} \to g(a, b) \leq f(a, b)]. \tag{di}$$

is necessary and sufficient for the discriminator to exist. This observation leads us to a definition of the class of dMIAs (denoted by **dMIA**) as composed of $PS$-algebras that satisfies (**di**). We will exhibit an axiom system for the variety generated by **dMIA** and we will show that it is a proper subvariety of $\mathbf{Eq}(\mathbf{wMIA})$.

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