The Priestley duality for $\prec$-distributive $\vee$-predomains

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In his celebrated paper [13], M. Stone gave a topological representation for Boolean algebras, linking the worlds of topology and lattices together for the first time. He showed that the category of Boolean algebras and lattice homomorphisms is dually equivalent to that of Stone spaces and continuous maps. Shortly afterwards, Stone [14] extended this topology-lattice duality for the larger class of bounded distributive lattices by introducing the nowadays known spectral spaces. With the aim of connecting lattices to the classical Hausdorff topological spaces, Priestley [11] used ordered topological spaces and proposed another topological representation, nowadays known as Priestley spaces, for bounded distributive lattices. She established a dual equivalence between the category of bounded distributive lattices with lattice homomorphisms and that of Priestley spaces with monotone continuous maps.

Duality theory between topology and lattices has been extensively applied in many fields such as topology, functional analysis and logic, among others, and the study of topological representations for general partially ordered structures has been attracting wide attention. Starting from the 1970s, Stone duality has been generalized for spatial frames [9, 10], semilattices [5] and other classes of posets [6, 8]. Indeed, Gräzter [6] removed binary infima from bounded distributive lattices and obtained a topological representation for bounded distributive join-semilattices. Hofmann and Lawson [8] developed a Stone duality for continuous frames. Moreover, scholars have extended Priestley duality to bounded lattices [15] and to other classes of posets [1, 2, 7]. In particular, Hansoul and Poussart [7] proposed a Priestley-type topological representation for bounded distributive sup-semilattices. Besides, there are some works obtained by restricting Priestley duality to subcategories of bounded distributive lattices and lattice homomorphisms. Both the Pultr-Sichler duality [12] for frames and the Bezhanishvili-Melzer duality [3] for continuous frames fall into this category, for instance.

Recently, Bice [4] unified distributive join-semilattices and continuous frames as $\prec$-distributive $\vee$-predomains and further developed a Stone duality for $\prec$-distributive $\vee$-predomains, which is a common extension of the Hofmann-Lawson duality and the Grätzer duality. Precisely, he proved that the category of locally compact sober spaces with a base closed for finite unions is equivalent to that of $\prec$-distributive $\vee$-predomains. However, a Priestley duality for $\prec$-distributive $\vee$-predomains is unknown, and Bice left it open in [4].

In this talk, we give an affirmative answer to the above question of Bice. We introduce DP-compact pospaces as follows.

Definition 1. We call a tuple $(X, \tau, \leqslant, X_1, \beta)$ *DP-compact pospace* if

(1) $(X, \tau, \leqslant)$ is a compact pospace,

(2) X_1 is dense and order generating,

(3) β is composed by admissible lower open sets and closed under finite unions,

(4) $x \in X_1$ if and only if $\{U \in \beta \mid x \in U\}$ is a neighborhood base of x with respect to (X, τ^b).

Here, a lower open set U is *admissible* if $U = \downarrow(U \cap X_1)$. τ^b is the topology consisting of all lower open sets in $(X, \tau, \leqslant)$.

Then we establish a one-to-one correspondence between $\prec$-distributive $\vee$-predomains and DP-compact pospaces. To develop a dual equivalence, we further propose DP-morphisms and $\prec$-morphisms. Let **DPCP** be the category of DP-compact pospaces with DP-morphisms and **DP** be the category of $\prec$-distributive $\vee$-predomains with $\prec$-morphisms. Then we obtain our main theorem:

Theorem 2 (Main theorem)**.** *Categories* **DP** *and* **DPCP** *are dually equivalent.*

In addition, we would like to stress that our results restrict to the Hansoul-Poussart duality [7] and a Priestley duality for continuous frames. The Priestley-type topological representations of continuous frames are CF-compact pospaces defined as follows.

Definition 3. We call a tuple $(X, \tau, \leqslant, X_1)$ a *CF-compact pospace* if

(1) $(X, \tau, \leqslant)$ is a compact pospace,

(2) X_1 is dense and order generating,

(3) $x \in X_1$ if and only if $\{U \in \tau^b \mid x \in U = \downarrow(U \cap X_1)\}$ is a neighborhood base of x with respect to (X, τ^b).

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