Polynomial time checking of generalized Sahlqvist shape

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in joint work with
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In classical normal modal logic, the class of Sahlqvist formulae has several desirable properties such as defining canonical logics [15, 16]. The proof of canonicity of the logics defined by Sahlqvist axioms is obtained by proving that Sahlqvist formulae are elementary, i.e., the classes of frames they define are also defined by first order sentences. Besides the original Sahlqvist-van Benthem algorithm to compute such *first order correspondent* for any given Sahlqvist formula, other algorithms for second order quantifier elimination have been adapted to Sahlqvist formulas, such as SCAN [9] and DLS [8, 14]. Sahlqvist formulas and their correspondents are interesting also from a proof-theoretic perspective: for instance, Negri has shown that analytic calculi can be effectively generated for all the modal logics in the Sahlqvist fragment [13], since the first order correspondents of Sahlqvist formulas are generalized geometric formulas.

In [11], Goranko and Vakarelov extended Sahlqvist canonicity and correspondence results to the class of *inductive formulas* (also known as *generalized Sahlqvist* formulas), which is strictly large than the class of Sahlqvist formulas. Based on SCAN and DLS, the algorithm SQEMA for correspondence on inductive formulas has been introduced in [1, 3].

By reframing Sahlqvist theory in algebraic terms, the syntactic notion Sahlqvist and inductive formulas have been imported as Sahlqvist and inductive *inequalities* in much more general settings, and correspondence and canonicity properties analogous to the classical ones have been proved [7, 2, 4, 6]. Such developments extend Sahlqvist theory to all the logics the algebraic semantics of which are given by (distributive) normal lattice expansions (LE), e.g., intuitionistic modal logic, positive modal logic, orthologic, the full Lambek calculus, the multiplicative-additive fragment of linear logic, semi De-Morgan logic, and so on. The Ackermann Lemma Based Algorithm (ALBA) has been introduced as a successor of SQEMA to compute first order correspondents in such a general setting. Similarly, also proof theoretic results concerning inductive inequalities in such logics which partially extend the classical ones have been proved in [12], and results proving canonicity in a constructive meta-theory reflecting [10] have been proved in [5].

Contrary to the classical case, checking whether a given inequality is inductive is not an obviously easy task. Indeed, the strong properties characterising the Boolean setting make it possible to define the class of Sahlqvist inequalities in a way that straightforwardly induces a polynomial-time algorithm (on the length of the formula) to check whether a formula belongs in this class. The definition of inductive (and Sahlqvist) inequality in the more general LE setting is more involved, and a naive approach would check a certain property (in polynomial time) for each strict order on the variables, and for each polarity (either positive or negative) assignment on the variables; hence it would have time complexity $\mathcal{O}(2^v v!p(n))$, where v is the number of variables in the inequality, n is the length of the formula, and p is a polynomial.

In this talk, we show an algorithm that computes whether an inequality is *refined inductive* in polynomial time, i.e., $\mathcal{O}(n+vl+v^2h+h^2)$, where n is the length of the formula, l the number of leaves in its syntax tree, v the number of variables in the inequality, h the number of topmost nodes having a certain property in the syntax tree.

Checking whether an inequality is inductive is equivalent to checking for the existence of a system of *refined inductive* inequalities which is semantically equivalent (relative to the appropriate class of LEs) to the given inequality. Since the algorithm ALBA for correspondence (on which most of the applications of inductive inequalities rely) pre-processes any input inductive inequality so as to obtain such a system of refined inductive inequalities, this algorithm finds its natural place in a practical implementation of ALBA in the preprocessing step applied to any input inequality.

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