SAT-universal CNF for Łukasiewicz logic

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The proof theory of multiple-valued logics, as well as its complexity, have been deeply studied, particularly for the class of the so-called fuzzy logics. However, the study of a systematic presentation of these logics with a view to the design of efficient satisfiability solvers has received less attention. Since satisfiability is usually the main logical question addressed in instances of real-world related problems, this study is motivated both from a purely mathematical and also a more applied perspective. Finding a clausal-form like definition that would help the automatic management of the SAT question is a rather open question, which we will address in this work. We will focus here in SAT as the problem of determining, for a given formula, whether there is an assignment making that formula true (sometimes called strong SAT), as opposed to other definitions related to assigning a particular value to the formula.

It is immediate that SAT for Gödel and Product logics is equal to that of classical logic (see eg. [3]), but the Łukasiewicz logic case offers deeper challenges. In the literature, we find studies on purely syntactical clausal forms for Łukasiewicz logics for instance in [4] and [2]. While the first one addresses only a subclass of Łukasiewicz formulas, the second offers a definition of a clausal form which is universal for SAT, but seems of limited use when attempting to design a resolution-like algorithm.

We propose a definition of clausal form for Łukasiewicz logic that is universal for SAT and whose structure offers a high potential, since the many-valued operators (namely, the non lattice ones) are applied to single literals.

Definition 1.1. We let monadic Łukasiewicz formulas be the formulas build with the language $\oplus, \odot$ and a single literal.

For instance, $((\neg x)^3 \oplus (\neg x)) \odot (3\neg x)$ is a monadic Łukasiewicz term, while $x \oplus y$ or $x \odot \neg x$ are not.[1]

Definition 1.2. A formula φ is in Ł-SAT conjunctive normal form if it has the structure

$$\bigwedge_{i\in I} \bigvee_{j\in J} t_{i,j}(x_{i,j})$$

for t monadic Łukasiewicz formulas.
We denote by Ł-SAT$_{CNF}$ to the set of formulas in Ł-SAT conjunctive normal form.

We can define a mapping $\sigma\colon Fm \to$Ł-SAT$_{CNF}$ in such a way that the following result holds:

Theorem 1.3. *Let φ be a Łukasiewicz formula. Then φ is SAT if and only if $\sigma(\varphi)$ is SAT.*

The proof and construction rely in several known results about Łukasiewicz logic, namely:

Lemma 1.4 (from [1])**.** *φ is SAT in L if and only if it is SAT in MV_n for some $n \leqslant (\frac{\sharp\varphi}{n})^n$, for $\sharp\varphi$ the number of apparitions of variables in φ and n the number of different variables in φ.*

[1] By l^n or nl we mean the usual application of the Łukasiewicz product or sum n times.

This implies that φ is SAT in Ł if and only if it is SAT in MV_{k_φ}, for $k_\varphi = mcm(\{p : p \leqslant \frac{\sharp\varphi}{n})^n, p \text{ prime}\})$.

Lemma 1.5 (Existence of so-called Ostermann terms, from [5])**.** *Let $a \in [0,1]$ be a finite sum of inverses of powers of 2. Then there is a formula from $[0,1]$ in $[0,1]$ in one free variable $\tau_a(x)$, such that*

1. *$\tau_a(x) = 1$ if and only if $x \geqslant a$,*
2. *$\tau_a(x)$ is a composition of $y \odot y$ and $y \oplus y$.*

We do not detail the construction of σ here for lack of space, but the sketch of the definition and proof of universality is as follows.

Let us denote by $\mathbb{D}$ the finite sums of inverse powers of 2 belonging to $[0,1]$, as in Lemma 1.5. It is easy to check that, given any n, we can chose some finite $\mathbb{D}_n \subset \mathbb{D}$ such that $0, 1 \in \mathbb{D}_n$ and for every $i/n, (i+1)/n \in MV_n$ (for $i < n$) there is a single $d_i \in \mathbb{D}_n$ for which $i/n < d_i < i+1/n$, and such that no other element belongs to $\mathbb{D}_n$. Furthermore, an involutive negation can be defined over them in the obvious way (namely, $\sim d_i = d_{n-i-1}$), as well as two suitable notions of (closed) product between them (roughly speaking, the top one, and the bottom one). Using these ideas, in combination with the above completeness for SAT with respect to a single finite algebra, we can define constructively the translation σ relying in the possibility to split each implication ($a \to b = 1$ if and only if, for any element x in $\mathbb{D}_{k_\varphi}$, either $a \leqslant x$ or $x \leqslant b$). The involutive negation (both over the elements of the algebra and over $\mathbb{D}_{k_\varphi}$), when used carefully, allows us to address both inequalities as the previous ones, leading to a total splitting of the formulas in Ostermann terms over the elements in $\mathbb{D}_{k_\varphi}$ applied to the literals arising from the variables in φ. The distributivity of MV algebras allows to conclude the final form as CNF.

We will also present a resolution method complete with respect to the presented forms, which needs of a finite number of rules to produce an assignment satisfying the formula. The fact that the outermost level is that of classical CNF, and that the multi-valuedness is limited to single variables makes this forms amenable to be solved either in the previous way or modeled with tools like MIP or relying, for the outermost level of the solving algorithm, in efficient classical SAT solvers. Furthermore, while the bound for finite satisfiability under Lemma 1.4 for the translated formula would be very high, a refinement of our Theorem, following from the proof itself, is that φ is SAT if and only if $\tau(\varphi)$ is SAT in MV_{k_φ}.

References

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