A Stone duality for the class of compact Hausdorff spaces

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Abstract

The Stone space of a boolean algebra is defined as the totally disconnected compact Hausdorff space whose points are the ultrafilters of the boolean algebra. Conversely, a totally disconnected compact Hausdorff space is homeomorphic to the Stone space of the boolean algebra of its clopen sets. Looking from a categorical perspective, the Stone duality can be expressed as the existence of a duality between the category of boolean algebras with arrows represented by adjoint homomorphisms and the category of totally disconnected compact Hausdorff spaces with continuous open maps. Furthermore, this duality restricts to one between the category of complete boolean algebras with complete homomorphisms and the category of extremally disconnected compact Hausdorff spaces with continuous open maps.

Generalizing this idea, we present an algebraic characterization of T_0-topological spaces in terms of preorders describing a base for the space. In particular, we show that any T_0-topological space can be represented as the space whose points are the neighborhood filters of one of its basis for the open sets. Conversely, we show that any dense family of filters on a preorder defines a topological space whose characteristics are strictly connected to the ones of the preorder. Therefore, we show how the separation properties of the topological space can be described in terms of the algebraic properties of the corresponding preorder and family of filters.

Furthermore, drawing on Orrin Frink's article [1], we outline the algebraic conditions on a selected base of the topological space ensuring that the space is compact and Hausdorff.

Indeed, in his article [1], Frink provided an internal characterization of Tychonoff spaces: specifically, he proved that a space is Tychonoff if and only if it admits a *normal* base for the closed sets of a space, i.e. a base that forms a disjoint ring of sets, where disjoint members can be separated by disjoint complements of members of the base. Moreover, he showed that if Z is a normal base for X, then the space of Z-*ultrafilters* forms a Hausdorff compactification for X. Clearly, we can obtain different compactifications of the same non-compact Tychonoff space by choosing different normal bases.

Following this approach, we characterize the algebraic properties that a preorder must possess in order to induce a Tychonoff space. In particular, we show that every Tychonoff space can be described as the space whose points are *some* minimal prime filters of a particular type of distributive lattices. Furthermore, we show that the space obtained considering *all* the prime minimal filters of it forms a Hausdorff compactification of the original Tychonoff space.

These results allows us to define a duality between the category of compact Hausdorff spaces with continuous maps and a suitable category of lattices.

This is joint work with Matteo Viale.

References

[1] Orrin Frink. Compactifications and semi-normal spaces. *American Journal of Mathematics*, 86(3):602–607, 1964.

[2] Matteo Viale. Notes on forcing, 2017. http://www.logicatorino.altervista.org/matteo_viale/DispenseTI2014.pdf.