# Temporal Logic of a Sequence of Finite Linear Processes

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We provide a sound and complete axiomatization of a temporal logic of a sequence of finitely many finite linear structures linked by surjective bounded morphisms.

Finite linear structures, *i.e.*, finite sets with a strict linear ordering, naturally arise as representations of a discrete, bounded time flow. Many domains of our everyday practice including time series [1], scene analysis [6], chain-of-responsibility design pattern in programming [3], [5], *etc.* involve a finite linear structure to represent a sequence of consecutive steps. A familiar example of such a structure is a movie represented as a sequence of individual frames.

In such scenarios, it is often natural to group consecutive elements into conceptually meaningful units in such a way that these units inherit the temporal order of the original structure. Moreover, this process can be repeated finitely many times. For a typical example of what is meant here consider a set of movie frames, grouped into episodes, these further grouped into scenes, which finally form acts. The structure of episodes inherits the temporal order from the ordering of individual frames. The same is true for the structure of scenes, and that of acts.

**Definition 1.** *A **TES** (Temporal Event Structure) is* $(F_1, \dots, F_n, <_1, \dots, <_n, f_1, \dots, f_{n-1})$ *where* $(F_i, <_i)$ *are finite strict linear orders, while* $f_i : F_i \twoheadrightarrow F_{i+1}$ *are onto monotone maps, where monotone means* $f_i(a) \leq_{i+1} f_i(b)$ *for all* $a \leq_i b$*. Let* $F \coloneqq \bigcup_{i=1}^{n} F_i$, $<\coloneqq \bigcup_{i=1}^{n} <_i$, $f \coloneqq \bigcup_{i=1}^{n-1} f_i$.

The language $\mathcal{L}$ is given by: $\phi = \ p \ \mid \ \neg\phi \ \mid \ \phi \wedge \phi \ \mid \ \Box_d \phi \ \mid \ \Box_u \phi \ \mid \ \Box_r \phi \ \mid \ \Box_l \phi$, where $p$ ranges over proposition symbols. Other logical symbols are defined as usual.

Our intention is to interpret the language $\mathcal{L}$ over an arbitrary **TES** $(F, <, f)$ in such a way that $\Box_r$ and $\Box_l$ range over $(F, <, >)$ while $\Box_d$ and $\Box_u$ range over $(F, f, f^{-1})$. Denote the class of all **TES** with $n$ fixed by $\mathbf{T_n}$. The logic $\mathsf{Log}(\mathbf{T_n})$ is the set of all formulas of $\mathcal{L}$ valid on all structures in $\mathbf{T_n}$. $\mathsf{Log}(\mathbf{T_2})$ was investigated in a recent paper [2].

In this contribution we present an axiomatization of $\mathsf{Log}(\mathbf{T_n})$ for arbitrary fixed $n > 1$. Let $\mathbf{L_n}$ be the least subset of $\mathcal{L}$ containing the following set of axioms and closed under the standard rules of uniform substitution, modus ponens and necessitation.

- All classical tautologies, standard axioms of modal logic **K** for each modal operator;

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|---|---|---|
| `Inv`:<br>$p \to \Box_l \Diamond_r p \wedge \Box_r \Diamond_l p$<br>$p \to \Box_u \Diamond_d p \wedge \Box_d \Diamond_u p$ | `GL`:<br>$\Box_r(\Box_r p \to p) \to \Box_r p$<br>$\Box_l(\Box_l p \to p) \to \Box_l p$ | `NoBranching`:<br>$\Diamond_r \Diamond_l p \to \Diamond_r p \vee p \vee \Diamond_l p$<br>$\Diamond_l \Diamond_r p \to \Diamond_r p \vee p \vee \Diamond_l p$ |
| `Level`:<br>$\bigwedge_{k=1}^{n-1} \left( \Box_u^k \bot \to \Diamond_d^{n-k} \top \right)$ | `Length`:<br>$\Box_d^n \bot$ | `Coherence`:<br>$\bigwedge_{k=1}^{n-1} \left( \Diamond_d^k \top \to \Box_r \Diamond_d^k \top \wedge \Box_l \Diamond_d^k \top \right)$ |
| `Surj`:<br>$\bigvee_{k=1}^{n-1} \left( \Diamond_u^k \top \to \Box_r \Diamond_u^k \top \wedge \Box_l \Diamond_u^k \top \right)$ | `Bounded`:<br>$\Diamond_d \Diamond_r p \to \Diamond_r \Diamond_d p$ | `DomConn`:<br>$\Diamond_d \Diamond_u p \to \Diamond_r p \vee p \vee \Diamond_l p$ |
| `Func`:<br>$p \to \Box_u \Box_d p$ | `Monot`:<br>$\Diamond_r \Diamond_d p \to \Box_d (p \vee \Diamond_r p)$ | |

Abstract Kripke semantics for $\mathcal{L}$ is provided by Kripke frames $\mathcal{F} = (W, R, R_f, R', R'_f)$ where $W$ is a nonempty set and each of $R, R_f, R', R'_f \subseteq W \times W$ is a binary relation.

**Definition 2.** *We will say that a frame $\mathcal{F} = (W, R, R_f, R', R'_f)$ is an $\mathbf{L_n}$-frame if the following conditions are satisfied: $W = \bigcup_{i=1}^{n} W_i$ where for all distinct $i, j \leq n$ we have $W_i \neq \emptyset$ and $W_i \cap W_j = \emptyset$; $R' = R^{-1}$; $R, R'$ are non-branching, transitive and well-founded and $R = \bigcup_{i=1}^{n} R_i$ where $R_i = R \cap (W_i \times W_i)$ for $i \leq n$; $R_f \cap (W_i \times W_{i+1})$ is a surjective bounded morphism with respect to $R_i$ and $R_{i+1}$; $R'_f = R_f^{-1}$ and $R_f$ is domain connected [2, Def. 3.6].*

**Theorem 3.** *For an arbitrary frame $\mathcal{F}$ it holds that $\mathcal{F} \models \mathbf{L_n}$ iff $\mathcal{F}$ is an $\mathbf{L_n}$-frame.*

Clearly a disjoint union of $\mathbf{L_n}$-frames is again an $\mathbf{L_n}$-frame. This implies that $\mathbf{L_n}$-frames can be infinite, and fail the trichotomy property for $R_i$, $i \leq n$, while our intended models, **TES**s are finite with $<_i$ trichotomous. To retain finiteness and trichotomy, we focus our attention on connected $\mathbf{L_n}$-frames, *i.e.* on $\mathbf{L_n}$-frames which cannot be presented as a disjoint union of two $\mathbf{L_n}$-frames. It turns out that a connected $\mathbf{L_n}$-frame is in a way isomorphic to a **TES**.

**Theorem 4.** *In every connected $\mathbf{L_n}$-frame $\mathcal{F} = (W, R, R_f, R', R'_f)$ the set $W$ is finite and each relation $R_i$ is trichotomous.*

The class of connected $\mathbf{L_n}$-frames is modally undefinable since it is not closed under disjoint unions. The next theorem links connected $\mathbf{L_n}$-frames and **TES**s.

**Theorem 5.** *There is a one-to-one correspondence between the class $\mathbf{T_n}$ and the class of all connected $\mathbf{L_n}$-frames.*

The next theorem shows that each **TES** can be fully described by an $\mathcal{L}$-formula.

**Theorem 6.** *Given a **TES** $\mathcal{F} = (F, <, f)$ there is a formula $\phi_{\mathcal{F}} \in \mathcal{L}$ such that for an arbitrary **TES** $\mathcal{T}$ we have: $\mathcal{T} \models \phi_{\mathcal{F}}$ iff $\mathcal{T}$ is isomorphic to $\mathcal{F}$.*

Finally, we establish our main finding:

**Theorem 7.** *The logic $\mathbf{L_n}$ is sound and complete w.r.t. the class $\mathbf{T_n}$.*

It follows that the logic $\mathbf{L_n}$ has the finite model property and is decidable.

**Acknowledgements:** The work has been supported by Shota Rustaveli National Science Foundation of Georgia grants #FR-22-6700 and #FR-22-4923.

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