The mathematical theory of contextuality: topological, algebraic, categorical and logical aspects

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Abstract

The key signature of non-classicality in quantum mechanics is contextuality. In classical physics, observable quantities have well-defined values independently of which measurements are performed. This is contradicted by the predictions of quantum mechanics, as verified in numerous experiments. These say that values can only be assigned locally, in measurement contexts, *i.e.* with respect to sets of measurements which can be performed together, providing observational “windows” of classical information on the quantum system. These windows may overlap, and will agree on their overlaps (*local consistency*), but it is not possible, on pain of logical contradiction, to glue all these pieces of information together (*global inconsistency*). Apart from the foundational significance of contextuality, implying a fundamental shift in our view of physical reality, it is a major source of quantum computational advantage.

Contextuality has a rich mathematical structure, which can be developed independently of quantum mechanics, providing powerful tools which can be applied to quantum computation and foundations, and far beyond. The same contextual structures can be found to arise naturally in classical computer science (database theory, constraint satisfaction), linguistics, and beyond.

The mathematical structures arising naturally in the study of contextuality include category theory, sheaf theory, and logic:

- contextuality can formulated in terms of obstructions to passing from local to global. The natural language for this is sheaf theory. The obstructions can be characterised using sheaf cohomology.
- Bell inequalities, providing fundamental tools for experimental verification of contextuality (as in the 2023 Nobel prize in Physics) can be characterized in terms of logical consistency conditions on events – Boole’s “conditions of possible experience”.
- There is a resource theory of contextuality based on this, and notions of simulation between contextual systems, using the underlying categorical structure.
- Another fundamental ingredient is the convex structure and geometry associated with the use of probabilistic models.
- Algebraic structures, such as partial Boolean algebras and groups with commutation relations, also play a central role.

The aim of this course is to give an introduction to this body of material, oriented to the interests and backgrounds of the TACL community. No background in quantum mechanics will be required.

List of topics to be covered:

- Basic introduction to the concept of contextuality, and how it arises in quantum mechanics
- functorial formulation of contextuality, using some (very basic) notions of sheaf theory
- cohomological characterization of contextuality
- logical theory of contextuality, partial Boolean algebras, duality theory for partial Boolean algebras
- Bell inequalities, and their characterization using logical consistency condition
- resource theory of contextuality, notions of simulation between contextual systems

References

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