The mathematical theory of contextuality
Lecture 1: Introduction

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TACL 2024 Summer School

People

Adam Brandenburger, Lucien Hardy, Shane Mansfield, Rui Soares Barbosa, Ray Lal, Mehrnoosh Sadrzadeh, Phokion Kolaitis, Georg Gottlob, Carmen Constantin, Kohei Kishida. Giovanni Caru, Linde Wester, Nadish de Silva, Martti Karvonen

References

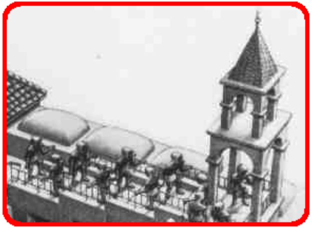
Some papers (all available on the arXiv):

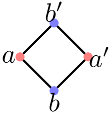
- *The sheaf-theoretic structure of non-locality and contextuality*, SA and Adam Brandenburger, (2011)
- *Logical Bell inequalities*, SA and Lucien Hardy (2012)
- *Contextual Semantics: From Quantum Mechanics to Logic, Databases, Constraints, and Complexity*, SA (2014)
- *Contextuality, cohomology and paradox*, SA, Rui Soares Barbosa, Kohei Kishida, Ray Lal and Shane Mansfield (2015)
- *The contextual fraction as a measure of contextuality*, SA, Rui Soares Barbosa and Shane Mansfield (2017)
- *Towards a complete cohomology invariant for non-locality and contextuality*, Giovanni Carù (2018)
- *The logic of contextuality*, SA and Rui Soares Barbosa (2021)

Contextuality in a nutshell

Where we have a family of data which is
locally consistent, but **globally inconsistent**

Contextuality Analogy: Local Consistency





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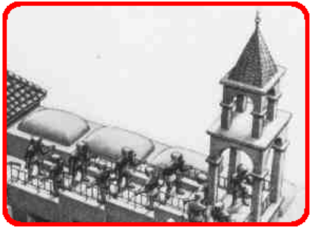
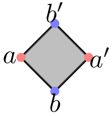
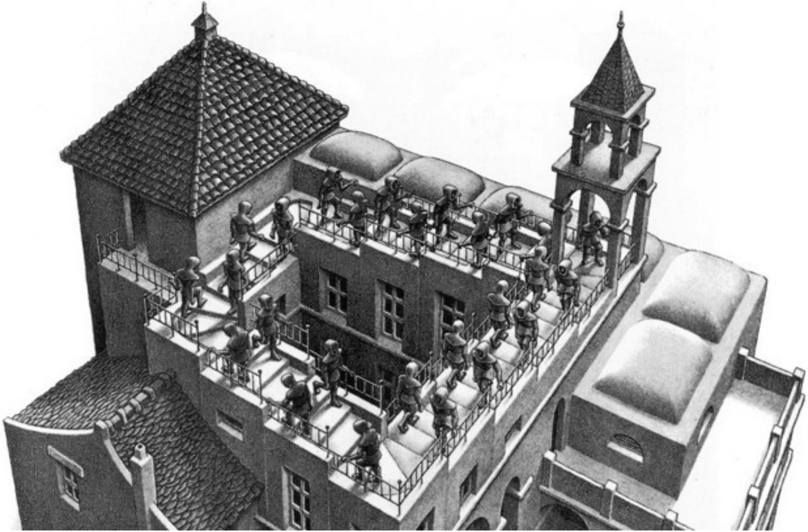


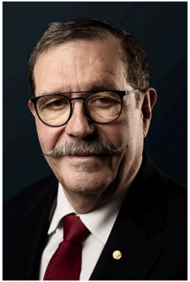


Contextuality Analogy: Global Inconsistency



The Nobel Prize in Physics 2022

Summary



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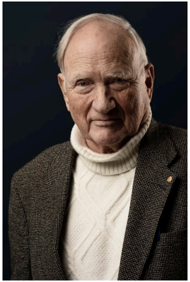
Alain Aspect

Prize share: 1/3



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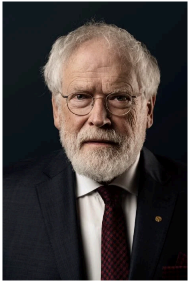
John F. Clauser

Prize share: 1/3



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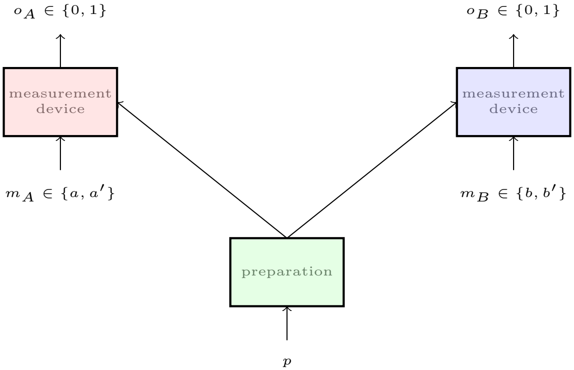
Anton Zeilinger

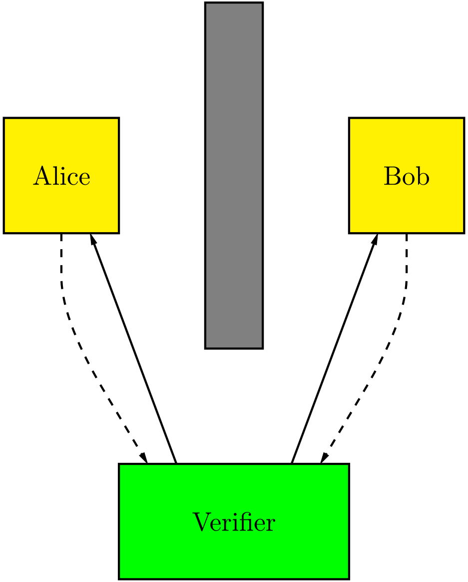
Prize share: 1/3

The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"

Testing non-local correlations



Alice-Bob games



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A table of conditional probabilities $p(a,b|x,y)$ defines a **probabilistic strategy** for this game. The **success probability** for this strategy is:

$$1/4[p(a=b|x=0,y=0) + p(a=b|x=0,y=1) + p(a=b|x=1,y=0)$$
$$+ p(a \neq b|x=1,y=1)]$$

A Strategy for the Alice-Bob game

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Example: The Bell Model

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
0	0	$1/2$	0	0	$1/2$
0	1	$3/8$	$1/8$	$1/8$	$3/8$
1	0	$3/8$	$1/8$	$1/8$	$3/8$
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The entry in row 2 column 3 says:

If the Verifier sends Alice a_1 *and Bob* b_2, *then with probability* 1/8, *Alice outputs a* 0 *and Bob outputs a* 1.

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If the Verifier sends Alice a_1 *and Bob* b_2*, then with probability* $1/8$*, Alice outputs a* 0 *and Bob outputs a* 1*.*

This gives a winning probability of $\frac{3.25}{4} \approx 0.81$.

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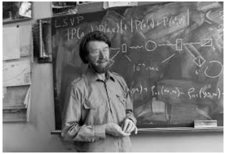
The Bell table exceeds this bound. Since it is **quantum realizable** using an entangled pair of qubits, it shows that quantum resources yield a **quantum advantage** in an information-processing task.

Logic and Probability: from Boole to Bell

George Boole 1815–64



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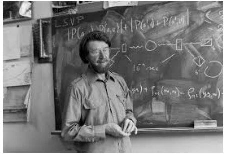

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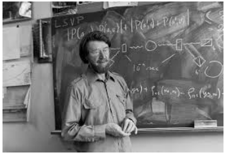

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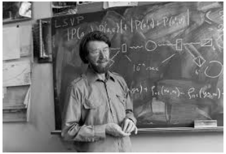

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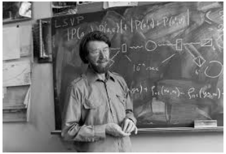

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Discussion in my paper *Classical Probability, Classical Logic, and Quantum Mechanics* in volume for Pitowsky *Quantum, Probability, Logic* (2020).

Boole's "conditions of possible experience"

Pitowsky's pellucid summary:

> *Boole's problem is simple: we are given rational numbers which indicate the relative frequencies of certain events. If no logical relations obtain among the events, then the only constraints imposed on these numbers are that they each be non-negative and less than one. If however, the events are logically interconnected, there are further equalities or inequalities that obtain among the numbers. The problem thus is to determine the numerical relations among frequencies, in terms of equalities and inequalities, which are induced by a set of logical relations among the events. The equalities and inequalities are called "conditions of possible experience".*

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More formally, we are given basic events $E_1, \ldots, E_n$, and boolean functions $\varphi_1, \ldots, \varphi_m$ of these events. Such a function can be described by a propositional formula in the variables $E_1, \ldots, E_n$.

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> Question: What **numerical** relationships between the probabilities can we infer from the **logical** relationships between the events?

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Using elementary probability theory, we can calculate:

$$p_N \;\leq\; \mathsf{Prob}(\bigvee_{i=1}^{N-1} \neg\phi_i) \;\leq\; \sum_{i=1}^{N-1} \mathsf{Prob}(\neg\phi_i) \;=\; \sum_{i=1}^{N-1} (1 - p_i) \;=\; (N-1) - \sum_{i=1}^{N-1} p_i.$$

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Hence we obtain the inequality

$$\sum_{i=1}^{N} p_i \;\leq\; N - 1.$$

Logical analysis of the Bell table

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If we read 0 as true and 1 as false, the highlighted entries in each row of the table are represented by the following propositions:

$$\begin{array}{lcccccccc}
\varphi_1 & = & (a_1 \wedge b_1) & \vee & (\neg a_1 \wedge \neg b_1) & = & a_1 & \leftrightarrow & b_1 \\
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The violation of the logical Bell inequality is $1/4$.

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Suppose that we have a K-consistent family $\{\varphi_i\}$ over the basic events $E_1, \ldots, E_n$. For any probability distribution on the set of truth-value assignments to the E_j, with induced probabilities $p(\varphi_i)$ for the events φ_i, we have:

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Remarkably, **all Bell inequalities arise this way** (Abramsky and Hardy, *Logical Bell inequalities*, Physical Review A 2012)

Theorem

A rational inequality is satisfied by all non-contextual empirical models if and only if it is equivalent to a logical Bell inequality of the above form.

Answering Boole, Quantum questions

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With contextuality, we are concerned with

quantum conditions of impossible experience

Science Fiction? – The News from Delft

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First Loophole-free Bell test, 2015

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日本語要約

Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres

B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau & R. Hanson

Nature **526**, 682–686 (29 October 2015) doi:10.1038/nature15759

Received 19 August 2015 Accepted 28 September 2015 Published online 21 October 2015

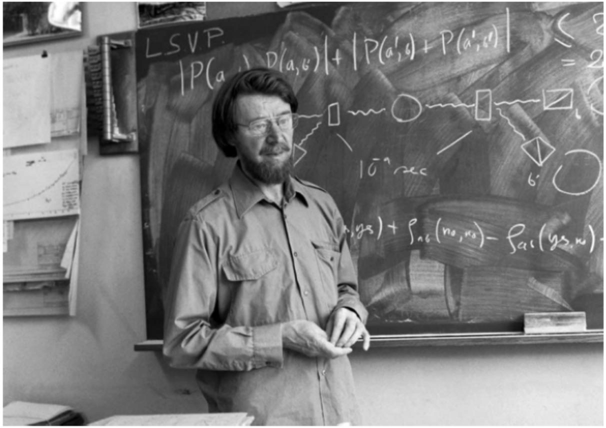
More than 50 years ago[1], John Bell proved that no theory of nature that obeys locality and realism[2] can reproduce all the predictions of quantum theory: in any local-realist theory, the correlations between outcomes of measurements on distant particles satisfy an inequality that can be violated if the particles are entangled. Numerous Bell inequality tests have been reported[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]; however, all experiments reported so far required additional assumptions to obtain a contradiction with local realism, resulting in 'loopholes'[13, 14, 15, 16]. Here we report a Bell experiment that is free of any such additional assumption and thus directly tests the principles underlying Bell's inequality. We use an event-ready scheme[17, 18, 19] that enables the generation of robust entanglement between distant electron spins (estimated state fidelity of 0.92 ± 0.03). Efficient spin read-out avoids the fair-sampling assumption (detection loophole[14, 15]), while the use of fast random-basis selection and spin read-out combined with a spatial separation of 1.3 kilometres ensure the required locality conditions[13]. We performed 245 trials that tested the CHSH–Bell inequality[20] $S \leq 2$ and found $S = 2.42 \pm 0.20$ (where S quantifies the correlation between measurement outcomes). A null-hypothesis test yields a probability of at most $P = 0.039$ that a local-realist model for space-like separated sites could produce data with a violation at least as large as we observe, even when allowing for memory[16, 21] in the devices. Our data hence imply statistically significant rejection of the local-realist null hypothesis. This conclusion may be further consolidated in future experiments; for instance, reaching a value of $P = 0.001$ would require approximately 700 trials for an observed $S = 2.4$. With improvements, our experiment could be used for testing less-conventional theories, and for implementing device-independent quantum-secure communication[22] and randomness certification[23, 24].

Quantum ‘spookiness’ passes toughest test yet

Experiment plugs loopholes in previous demonstrations of 'action at a distance', against Einstein's objections — and could make data encryption safer.

Zeeya Merali

27 August 2015



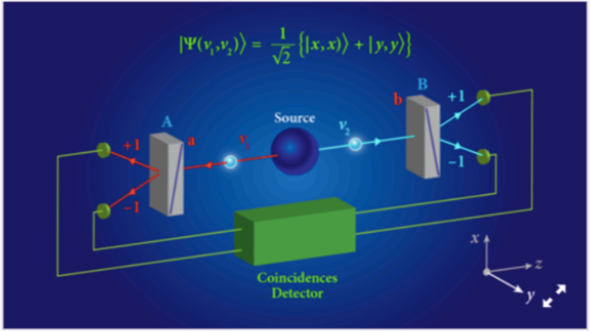
CERN

Viewpoint: Closing the Door on Einstein and Bohr's Quantum Debate

Alain Aspect, Laboratoire Charles Fabry, Institut d'Optique Graduate School, CNRS, Université Paris-Saclay, Palaiseau, France

December 16, 2015 • *Physics* 8, 123

By closing two loopholes at once, three experimental tests of Bell's inequalities remove the last doubts that we should renounce local realism. They also open the door to new quantum information technologies.



APS/Alan Stonebraker

Figure 1: An apparatus for performing a Bell test. A source emits a pair of entangled photons ν_1 and ν_2. Their polarizations are analyzed by polarizers A and B (grey blocks), which are aligned, respectively,

Timeline

1932	von Neumann's Mathematical Foundations of Quantum Mechanics
1935	EPR Paradox, the Einstein-Bohr debate
1964	Bell's Theorem
1982	First experimental test of EPR and Bell inequalities (Aspect, Grangier, Roger, Dalibard)
1984	Bennett-Brassard quantum key distribution protocol
1985	Deutch Quantum Computing paper
1993	Quantum teleportation (Bennett, Brassard, Crépeau, Jozsa, Peres, Wooters)
1994	Shor's algorithm
2015	First loophole-free Bell tests (Delft, NIST, Vienna)
2019	Quantum supremacy claimed by Google
2020	Quantum supremacy via boson sampling by USTC
2022	Nobel Prize in Physics for Aspect, Clauser and Zeilinger for Bell experiments
2024	Emerging quantum computing and technology industry ...

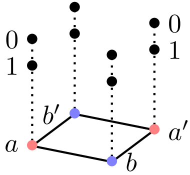
Formalising empirical data*

*SA, Brandenburger, *New Journal of Physics*, 2011.

in\out	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(a, b)	–	–	–	–
(a, b')	–	–	–	–
(a', b)	–	–	–	–
(a', b')	–	–	–	–

A **measurement scenario** $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- X – a finite set of measurements
- Σ – a simplicial complex on X
  faces are called the **measurement contexts**
- $O = (O_x)_{x \in X}$ – for each $x \in X$ a finite non-empty set of possible outcomes O_x



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An **empirical model** $e = \{e_\sigma\}_{e \in \Sigma}$ on $\mathbf{X}$:

- Each e_σ is a prob. distribution over joint outcomes $\prod_{x \in \sigma} O_x$ for σ
- **generalised no-signalling** holds: $\forall \sigma, \tau \in \Sigma, \sigma \subseteq \tau$.

$$e_\tau|_\sigma = e_\sigma$$

(i.e. marginals are well-defined)

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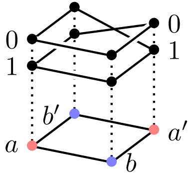

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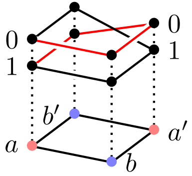

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The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.

Bundle Diagrams

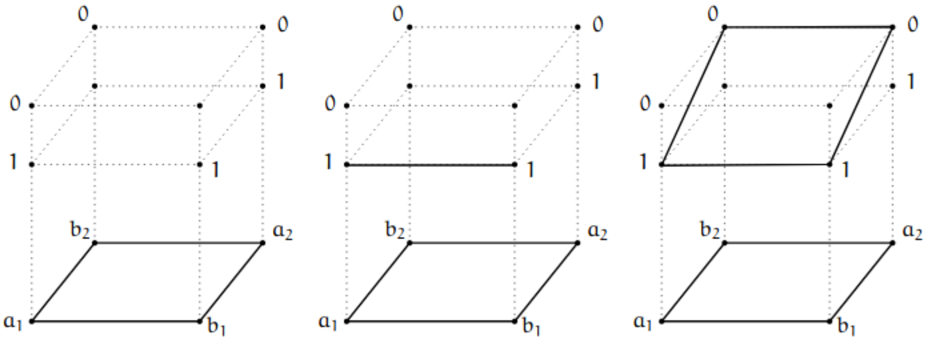

Figure 1: A (2,2,2) Bell-type scenario. The section $(a_1, b_1) \mapsto (1,1)$ is represented in the centre. On the right, the global section $(a_1, b_1, a_2, b_2) \mapsto (1,1,0,0)$

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
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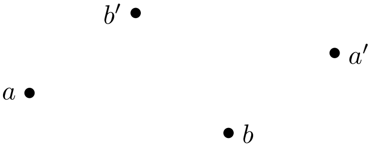

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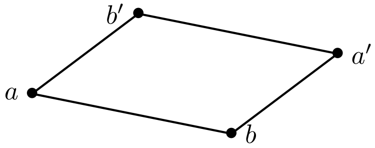

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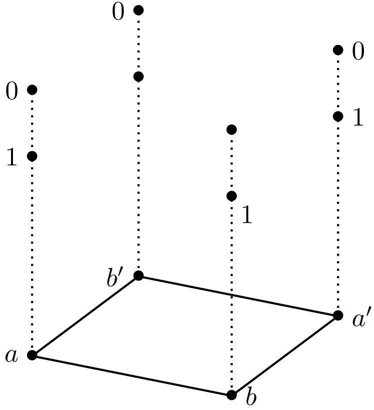

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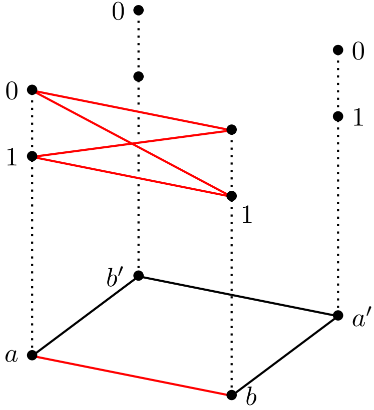

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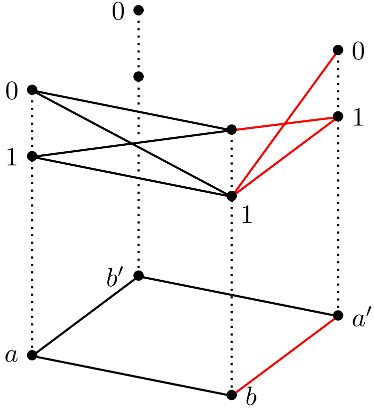

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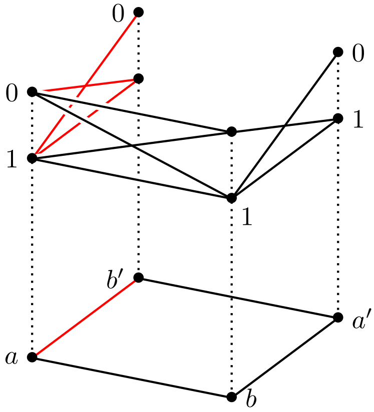

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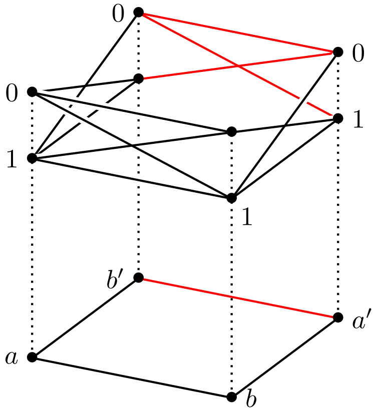

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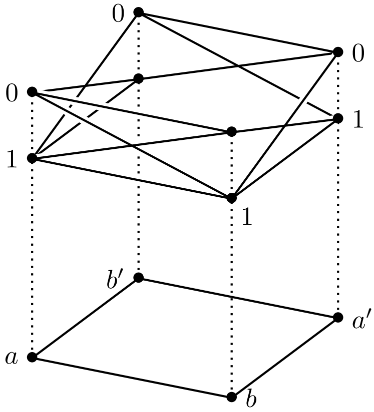

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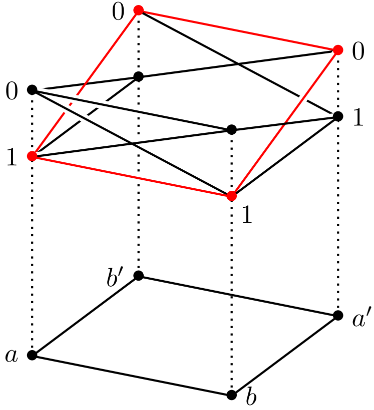

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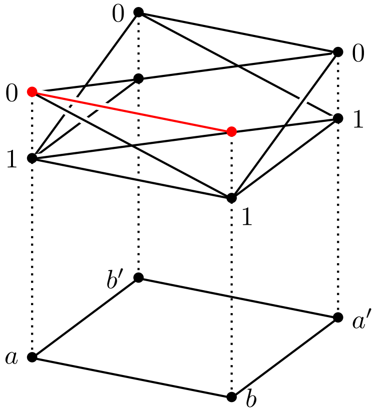

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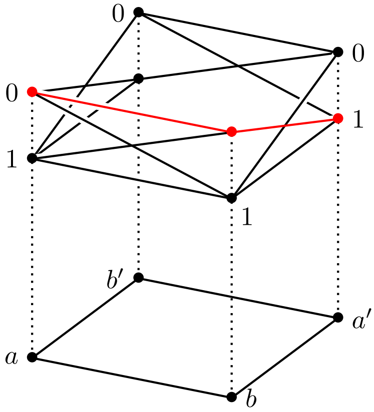

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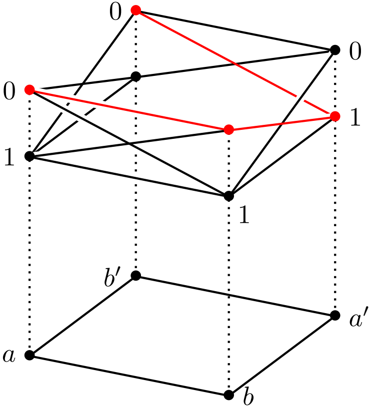

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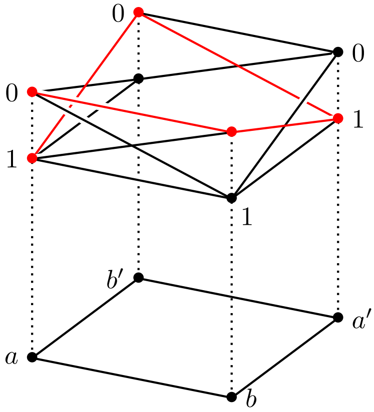

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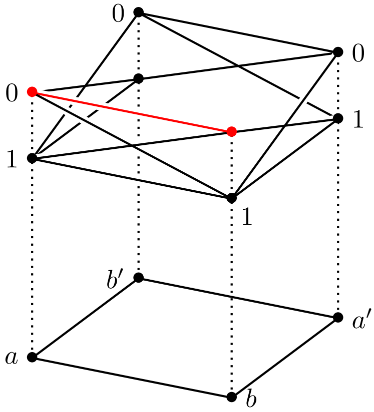

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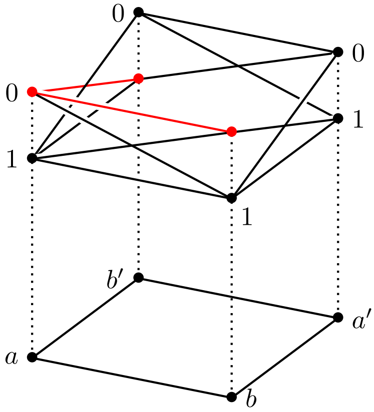

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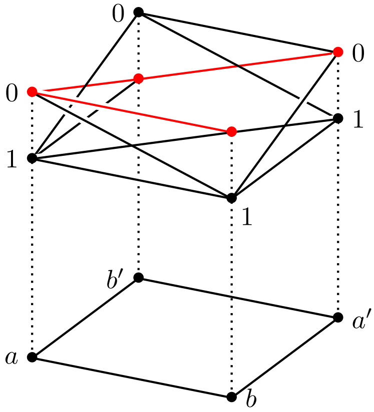

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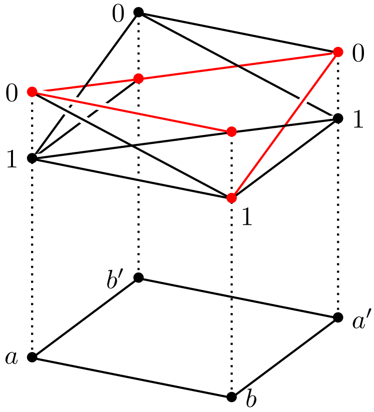


Contextuality

Definition

There is a hierarchy of contextuality

$$\text{Probabilistic} \subset \textbf{Logical} \subset \textbf{Strong}$$

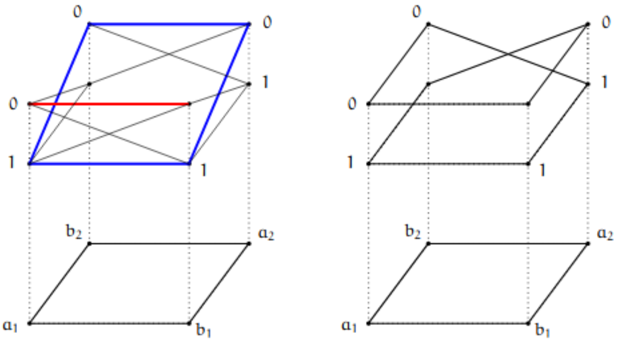
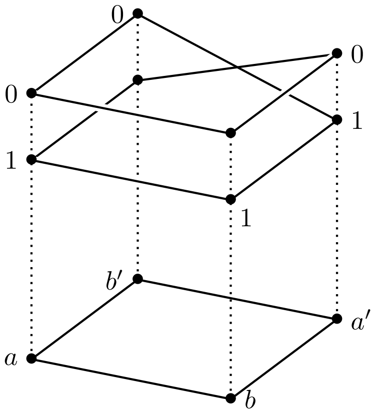


Figure 2: The Hardy model and the PR-Box model as bundle diagrams.

The Bell table and the “Möbius strip”

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a_1,b_1)	$1/2$	0	0	$1/2$
(a_1,b_2)	$3/8$	$1/8$	$1/8$	$3/8$
(a_2,b_1)	$3/8$	$1/8$	$1/8$	$3/8$
(a_2,b_2)	$1/8$	$3/8$	$3/8$	$1/8$



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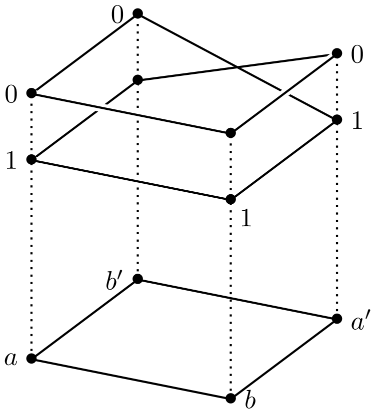
(a_2, b_2)	$1/8$	$3/8$	$3/8$	$1/8$



Physics $\leftrightsquigarrow$ Probability $\leftrightsquigarrow$ Logic $\leftrightsquigarrow$ Topology

Contextuality, Logic and Paradoxes

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Liar cycles. A Liar cycle of length N is a sequence of statements

$$\begin{array}{rl} S_1: & S_2 \text{ is true,} \\ S_2: & S_3 \text{ is true,} \\ & \vdots \\ S_{N-1}: & S_N \text{ is true,} \\ S_N: & S_1 \text{ is false.} \end{array}$$

For $N = 1$, this is the classic Liar sentence

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Following Cook, Walicki et al. we can model the situation by boolean equations:

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Contextuality, Logic and Paradoxes

Liar cycles. A Liar cycle of length N is a sequence of statements

$$\begin{aligned} S_1 &: \; S_2 \text{ is true,} \\ S_2 &: \; S_3 \text{ is true,} \\ &\vdots \\ S_{N-1} &: \; S_N \text{ is true,} \\ S_N &: \; S_1 \text{ is false.} \end{aligned}$$

For $N = 1$, this is the classic Liar sentence

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The "paradoxical" nature of the original statements is now captured by the inconsistency of these equations.

Contextuality in the Liar; Liar cycles in the PR Box

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We can regard each of these equations as fibered over the set of variables which occur in it:

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\{x_2, x_3\}: & x_2 & = & x_3 \\
 & \vdots & & \\
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Any subset of up to $n-1$ of these equations is consistent; while the whole set is inconsistent.

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The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.

The Robinson Consistency Theorem

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A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

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A minimal counter-example is provided at the propositional level by the following **Specker triangle**:

$$T_1 = \{x_1 \longrightarrow \neg x_2\},\ T_2 = \{x_2 \longrightarrow \neg x_3\},\ T_3 = \{x_3 \longrightarrow \neg x_1\}.$$