The mathematical theory of contextuality
Lecture 3: Quantum realizability

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Categorically, the category of matrices is a monoidal (even compact closed) skeleton of the category of finite-dimensional Hilbert spaces.

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If we represent qubit space with a standard basis $\{|0\rangle, |1\rangle\}$, then n-qubit space has basis

$$\{|s\rangle \ : \ s \in \{0,1\}^n\}$$

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- The possible outcomes of an observable $A = \sum_i \lambda_i P_i$ are given by the eigenvalues λ_i.
- The probability of getting the outcome λ_i when measuring A on the state Q represented by $|\psi\rangle$ is given by the **Born rule**:

$$\mathsf{Tr}(P_i Q) \;=\; |\langle e_i \,|\, \psi \rangle|^2$$

  where e_i represents the rank-1 projector P_i.

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We shall stick to the simplest level of presentation . . .

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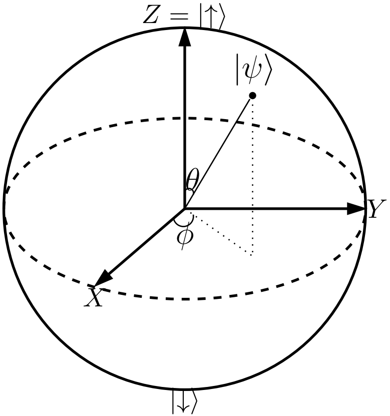
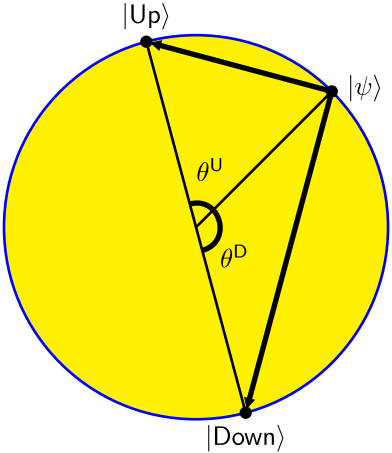
Developments such as **device-independent QKD**.

The Bloch sphere representation of qubits



Truth makes an angle with reality



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- Each pair $(\mathsf{Up}, \mathsf{Down})$ of antipodal points on the sphere define a possible measurement that we can perform on the qubit. Each such measurement has two possible outcomes, corresponding to Up and Down in the given direction. We can think of this physically e.g. as measuring Spin Up or Spin Down in a given direction in space.

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- When we subject a qubit to a measurement $(\mathsf{Up}, \mathsf{Down})$, the state of the qubit determines a probability distribution on the two possible outcomes. The probabilities are determined by the **angles** between the qubit state $|\psi\rangle$ and the points $(|\mathsf{Up}\rangle, |\mathsf{Down}\rangle)$ which specify the measurement. In algebraic terms, $|\psi\rangle$, $|\mathsf{Up}\rangle$ and $|\mathsf{Down}\rangle$ are unit vectors in the complex vector space $\mathbb{C}^2$, and the probability of observing Up when in state $|\psi\rangle$ is given by the square modulus of the inner product:

$$|\langle\psi|\mathsf{Up}\rangle|^2.$$

  This is known as the **Born rule**. It gives the basic predictive content of quantum mechanics.

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But in addition, we have the important feature that there are a continuum of possible questions we can ask. However, note that on each run of the system, we can only ask **one** of these questions. We cannot simultaneously observe Up or Down in two different directions. Note that this corresponds to the feature of the scenario we discussed, that Alice and Bob could only look at one their local registers on each round.

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Note in addition that a measurement has an **effect** on the state, which will no longer be the original state $|\psi\rangle$, but rather one of the states Up or Down, in accordance with the measured value.

Quantum Entanglement

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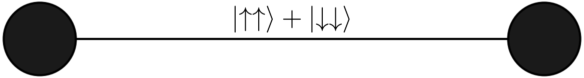
Bell state:



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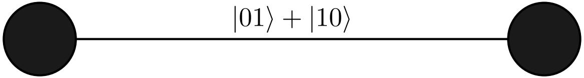

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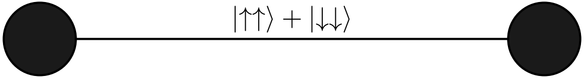

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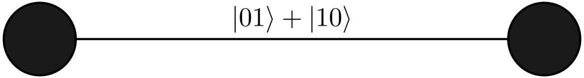

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Superposition encodes **correlation**.

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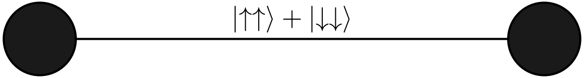

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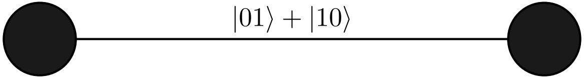

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Einstein's 'spooky action at a distance'. Even if the particles are spatially separated, measuring one has an effect on the state of the other.

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Example: The Bell Model

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
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Important note: this is **physically realizable**!

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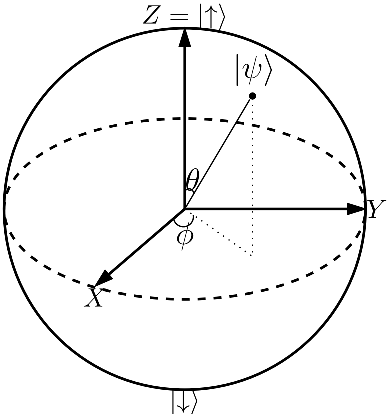

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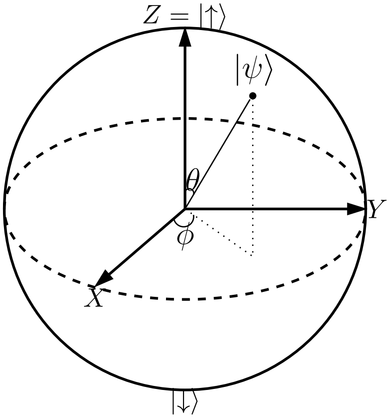

Spin measurements lying in the equatorial plane of the Bloch sphere
Spin Up: $(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle)/\sqrt{2}$, Spin Down: $(|\uparrow\rangle + e^{i(\phi+\pi)}|\downarrow\rangle)/\sqrt{2}$

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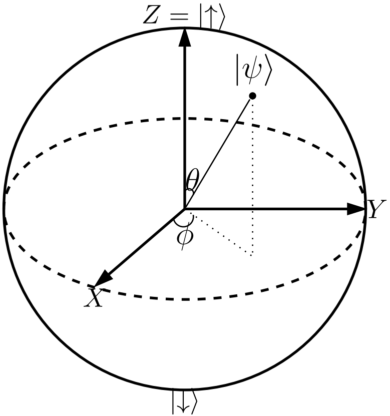

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Spin Up: $(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle)/\sqrt{2}$, Spin Down: $(|\uparrow\rangle + e^{i(\phi+\pi)}|\downarrow\rangle)/\sqrt{2}$

X itself, $\phi = 0$:
Spin Up $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and Spin Down $(|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$.

Computing the Bell table

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The event in yellow is represented by

$$\frac{|{\uparrow}\rangle + |{\downarrow}\rangle}{\sqrt{2}} \otimes \frac{|{\uparrow}\rangle + e^{i4\pi/3}|{\downarrow}\rangle}{\sqrt{2}} = \frac{|{\uparrow\uparrow}\rangle + e^{i4\pi/3}|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle + e^{i4\pi/3}|{\downarrow\downarrow}\rangle}{2}.$$

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Probability of this event M when measuring (a, b') on $B = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ is given by Born rule:

$$|\langle B|M\rangle|^2.$$

Computing Bell by Born

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Since the vectors $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ are pairwise orthogonal, $|\langle B|M\rangle|^2$ simplifies to

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The other entries can be computed similarly.

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Other attempts by Masanes and Mueller, Brukner and Dakic, the Pavia group (D'Ariano, Chiribella and Perinotti), . . .

Empirical Models

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Example: The Bell Model

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Important note: this is **quantum realizable**.

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Important note: this is **quantum realizable**.

Generated by Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

subjected to measurements in the XY-plane, at relative angle $\pi/3$.

The PR Box

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
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The PR Box

This satisfies No-Signalling, so is consistent with SR, but it is **not** quantum realisable.

Empirical models as vectors

We can regard an empirical model $\{d_C\}_{C \in \mathcal{M}}$ as a vector

$$\mathbf{v} \;=\; (\mathbf{v}_{C,s})_{C \in \mathcal{M}, s \in \mathcal{E}(C)}, \qquad \mathbf{v}_{C,s} := d_C(s)$$

in a high-dimensional real vector space.

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Note also that empirical models over a given measurement scenario are closed under convex combinations:

$$\mu d + (1-\mu)d')_C(s) \;:=\; \mu d_C(s) + (1-\mu)d'_C(s).$$

Moreover, convex combinations of compatible models are compatible.

The Quantum Set

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A subtle convex set sandwiched between two polytopes.

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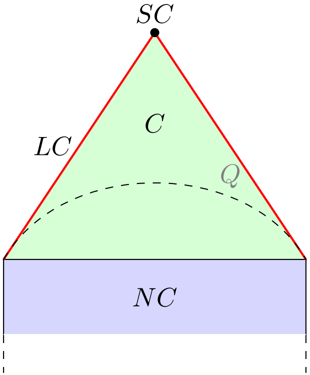

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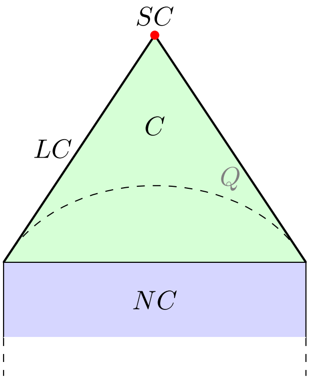
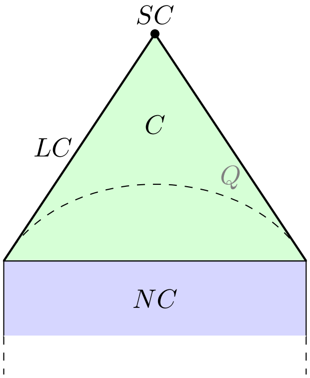


Key question: find compelling principles to explain why Nature picks out the quantum set.

Geometry of Empirical Models

For any given measurement scenario:



Geometry of Empirical Models

For any given measurement scenario:

(Probabilistic) Contextuality: relative interior
Logical Contextuality:
faces
Strong Contextuality:
Lower dimensional subspaces
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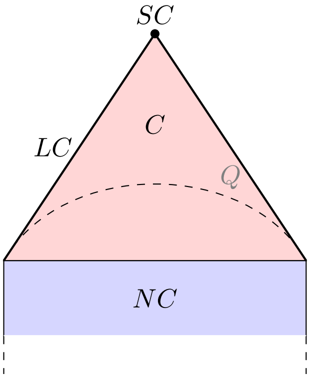
AvN $\subsetneq$ SC



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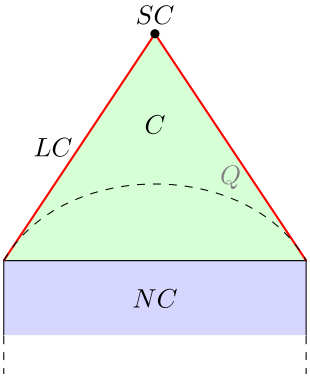

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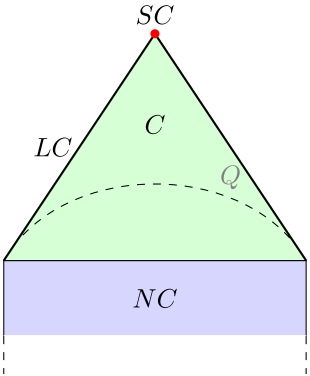

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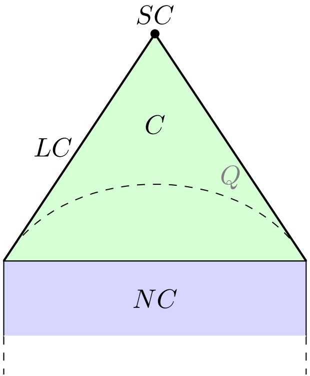

Probabilistic < Logical < Strong < AvN

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- Even more spectacularly, we have the $\mathsf{MIP}^* = \mathsf{RE}$ result of Ji, Natarajan, Vidick, Wright, Yuen (2020).
- This is simultaneously a major result in complexity theory, quantum foundations, and mathematics:
  - While $\mathsf{QIP} = \mathsf{IP} = \mathsf{PSPACE}$, allowing multiple quantum provers sharing entangled states allows all semidecidable problems to be represented (e.g. halting problem, provability of arithmetical statements).
  - The Tsirelson conjecture is refuted (in infinite dimensions). Commuting subalgebras cannot be represented on tensor products in general.
  - The Connes Embedding Problem is answered in the negative.

Quantifying contextuality: the contextual fraction

We look for a convex decomposition

$$e = \lambda e^{NC} + (1 - \lambda) e' \tag{1}$$

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1. Computable by a linear program.
2. The normalised violation by e of any Bell inequality is at most $\mathsf{CF}(e)$;
3. this bound is attained, *i.e.* there exists a Bell inequality whose normalised violation by e is $\mathsf{CF}(e)$;
4. moreover, for any decomposition of the form $e = \mathsf{NCF}(e)e^{NC} + \mathsf{CF}(e)e^{SC}$, this Bell inequality is tight at the non-contextual model e^{NC} and maximally violated at the strongly contextual model e^{SC}.

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Given a measurement scenario $\langle X, \mathcal{M}, O \rangle$, the **incidence matrix M** has

- rows indexed by $\langle C, s \rangle$, $C \in \mathcal{M}$, $s \in O^C$
- columns indexed by global assignments $g \in O^X$

$$\mathbf{M}[\langle C, s \rangle, g] := \begin{cases} 1 & \text{if } g|_C = s \\ 0 & \text{otherwise} \end{cases} \quad .$$

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Computing the non-contextual fraction corresponds to solving the following linear program:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\,\mathbf{c} \leq \mathbf{v}^e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \quad . \end{array} \tag{2}$$

Generalized Bell Inequalities

An **inequality** for a measurement scenario $\langle X, \mathcal{M}, O\rangle$ given by a set of coefficients $\alpha = \{\alpha(C,s)\}_{C\in\mathcal{M}, s\in\mathcal{E}(C)}$ and a bound R. For a model e, the inequality reads as

$$\mathcal{B}_\alpha(e) \ \leq \ R \ ,$$

where the left-hand side is given by

$$\mathcal{B}_\alpha(e) \ := \sum_{C\in\mathcal{M}, s\in\mathcal{E}(C)} \alpha(C,s) e_C(s) \ .$$

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Whereas a Bell inequality establishes a bound for the value of $\mathcal{B}_\alpha(e)$ amongst non-contextual models, for a general no-signalling model e, this quantity is limited only by

$$\|\alpha\| := \sum_{C \in \mathcal{M}} \max \{\alpha(C,s) \mid s \in \mathcal{E}(C)\}$$

Relating Bell inequality violation to the contextual fraction

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Let e be an empirical model. Then there is a Bell inequality whose normalised violation by e is exactly $\mathsf{CF}(e)$. Moreover, this Bell inequality is tight at the non-contextual model e^{NC}.

Quantifying Contextuality & Bell Inequalities

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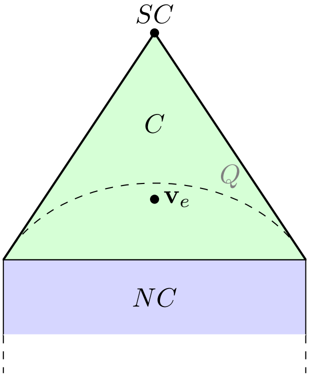

`maximise`	$\mathbf{1} \cdot \mathbf{x}$
`subject to`	$\mathbf{M}\,\mathbf{x} \leq \mathbf{v}_e$
`and`	$\mathbf{x} \geq \mathbf{0}$

Setting $\lambda = \mathbf{1} \cdot \mathbf{x}^*$

$$e = \lambda e_{\mathrm{NC}} + (1 - \lambda) e_{\mathrm{SC}}$$

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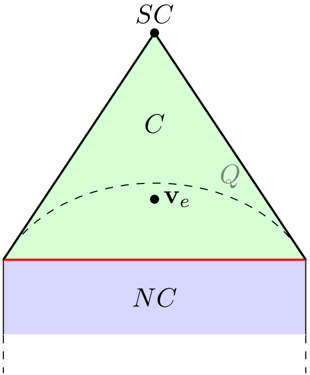

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Dual program:

$$\begin{array}{ll} \texttt{minimise} & \mathbf{y}\cdot\mathbf{v}_e \\ \texttt{subject to} & \mathbf{M}^T\,\mathbf{y} \ge \mathbf{1} \\ \texttt{and} & \mathbf{y} \ge \mathbf{0} \end{array}$$

computes tight Bell inequality (separating hyperplane)

Contextuality and quantum advantage

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- Measurement-based quantum computation (MBQC)
  - Raussendorf, *Physical Review A*, 2018.
  - SA, Barbosa, Mansfield, *Physical Review Letters*, 2018.

$$\overbrace{1-\bar{p}_S}^{\text{error}} \;\geq\; \underbrace{[1-\mathsf{CF}(e)]}_{\text{classicality}} \; \overbrace{\nu(f)}^{\text{hardness}}$$

quantifiable relationship!

The same quantitative relationship arises for

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Where the "line in the sand" is drawn separating quantum advantage from efficient classical simulability is still unclear.

Contextuality and quantum advantage with shallow circuits

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- With a two-stage query construction, this works for any choice of measurements. For the case of Weyl operators, a one-stage construction a la BGK is recovered.
- This provides a basis for a broader study of how to transform contextuality arguments systematically into instances of quantum advantage. Other promising areas where these ideas can be applied are communication complexity, and VQE solvers.