# Non-Classical Temporal Logic in Topological Dynamics

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DAY 1

# Topological Dynamics

Dynamical systems are abstract models of change over time and occur in many branches of mathematics and natural science.

Formally, a **dynamical (topological) system** is a pair $(X, S)$, where $X$ is a topological space and $S\colon X \to X$ is continuous.

We think of $X$ as representing **space** and $S$ as representing the passage of **time**.

A point $x \in X$ 'moves' along its orbit

$$x, S(x), S^2(x), \ldots, S^n(x), \ldots$$

**Recall:**
A topological space is a pair $(X, \mathcal{T})$ where $\mathcal{T} \subseteq 2^X$ satisfies

1. $\varnothing, X \in \mathcal{T}$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions

Elements of $\mathcal{T}$ are called **open sets**.

Sometimes omit mention of $\mathcal{T}$ and denote topological spaces by $X$.

# INTERIOR OPERATOR

If $X$ is a topological space and $A \subseteq X$, define

$$A^\circ = \bigcup \{U \subseteq A : U \text{ is open}\}$$

The set $A^\circ$ is the **interior** of $A$.

It is the largest open subset of $A$.

# Examples of Topological Spaces

- The real line $\mathbb{R}$ is equipped with its **standard topology** where $U \subseteq \mathbb{R}$ is open iff

$$\forall x \in U \exists \varepsilon > 0 \forall y \in \mathbb{R} \left( |x - y| < \varepsilon \Rightarrow y \in U \right)$$

- The rational numbers, $\mathbb{Q}$, are similarly equipped with the interval topology.

- For any $n$, $\mathbb{R}^n$ has a standard topology generated by **open balls**

$$B_\varepsilon(x) = \{y \in \mathbb{R}^n : d(x, y) < \varepsilon\}$$

## DEFINITION

A topological space $(X, \mathcal{T})$ is **Alexandroff** if whenever $\mathcal{U} \subseteq \mathcal{T}$, it follows that $\bigcap \mathcal{U}$ is open.

If $(W, \preccurlyeq)$ is a partially ordered set, then $W$ can be endowed with the **up-set topology** by letting $U \subseteq W$ be open if

$$\forall w \preccurlyeq v \left( w \in U \Rightarrow v \in U \right).$$

The **down-set** topology is defined dually.

## THEOREM

*A space $X$ is Alexandroff iff it is the up-set topology for some partial order $\preccurlyeq$ on $\mathcal{T}$.*

# CONTINUOUS FUNCTIONS

$S\colon \mathbb{R} \to \mathbb{R}$ is **continuous** if

$$\forall x \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 \left( |x - y| < \delta \Rightarrow |S(x) - S(y)| < \varepsilon \right)$$

More generally, $S\colon X \to Y$ is **continuous** if $U \subseteq Y$ is open $\Rightarrow$ $S^{-1}(U)$ is open.

If moreover $S(U)$ is open whenever $U$ is open, we say $S$ is an **interior map.**

**Examples**

- $S\colon \mathbb{R} \to \mathbb{R}$ is continuous whenever $S$ is a polynomial.
- If $(W, \preccurlyeq)$ is a preorder then $S\colon W \to W$ is continuous iff monotone:

$$\forall v, w \left( w \preccurlyeq v \Rightarrow S(w) \preccurlyeq S(v) \right)$$

# Poincaré Recurrence

A dynamical system $(X, S)$ is **probability-preserving** if for all open $A \subseteq X$, $|A| = |S^{-1}(A)|$, where $|A|$ denotes probability (or volume).

A dynamical system $(X, S)$ is **Poincaré recurrent** (for our purposes) if whenever $A$ is non-empty and open there are $x \in A$ and $n > 0$ such that $S^n(x) \in A$.

THEOREM (POINCARÉ)

*Every probability-preserving system where non-empty opens have positive probability is Poincaré recurrent.*

EXAMPLE: Rotation of a disk.

# Minimal Systems

A dynamical system $(X, S)$ is **minimal** if it contains no proper, closed, $S$-invariant subsystems.

PROPOSITION
*A dynamical system $(X, S)$ is minimal if and only if whenever $A$ is non-empty and open and $x \in X$, there is $n > 0$ such that $S^n(x) \in A$.*

THEOREM (BIRKHOFF)
*Every dynamical system on a compact space contains a (non-empty) minimal subsystem.*

EXAMPLE: **Irrarional** rotation of a circle.

# (Classical) Linear Temporal Logic

**Language ($\mathcal{L}_{\text{LTL}}$):**

$$p \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \psi \quad | \quad \circ\varphi \quad | \quad \Box\varphi$$

**Models:** $(X, S, [\![\cdot]\!])$ consisting of a set $X$, $S \colon X \to X$, and a **valuation** $[\![\cdot]\!] : \mathcal{L}_{\text{LTL}} \to 2^X$ such that

- $[\![p]\!] \subseteq X$ is any set
- $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\varphi]\!]$
- $[\![\circ\varphi]\!] = S^{-1}[\![\varphi]\!]$ (next)
- $[\![\Box\varphi]\!] = \bigcap_{n\in\mathbb{N}} S^{-n}[\![\varphi]\!]$ (henceforth)

# Intuitionistic Propositional Logic

**Language ($\mathcal{L}_0$):**

$$\bot \quad | \quad p \quad | \quad \varphi \wedge \psi \quad | \quad \varphi \vee \psi \quad | \quad \varphi \to \psi$$

**Kripke Models:** $(X, \preccurlyeq, [\![\cdot]\!])$ consisting of poset equipped with a valuation $[\![\cdot]\!] : \mathcal{L}_0 \to 2^X$ such that

- $[\![\bot]\!] = \varnothing$
- $[\![p]\!]$ is any **upward-persistent** set

$$(v \preccurlyeq w \in [\![p]\!] \Rightarrow v \in [\![p]\!])$$

- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$
- $w \in [\![\varphi \to \psi]\!] \Leftrightarrow \forall v \preccurlyeq w \, (v \in [\![\varphi]\!] \Rightarrow v \in [\![\psi]\!])$

# Topological Semantics

**Kripke Models:** $(X, \mathcal{T}, \llbracket\cdot\rrbracket)$ consisting of topological space equipped with a valuation $\llbracket\cdot\rrbracket : \mathcal{L}_0 \to \mathcal{T}$ such that

- $\llbracket\bot\rrbracket = \varnothing$
- $\llbracket p \rrbracket$ is any ~~**upward-persistent** set~~ open set
- $\llbracket\varphi \wedge \psi\rrbracket = \llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket$
- $\llbracket\varphi \vee \psi\rrbracket = \llbracket\varphi\rrbracket \cup \llbracket\psi\rrbracket$
- $w \in \llbracket\varphi \to \psi\rrbracket \Leftrightarrow$ ~~$\forall v \preccurlyeq w \, (v \in \llbracket\varphi\rrbracket \Rightarrow v \in \llbracket\varphi\rrbracket)$~~

  There is a neighbourhood $U$ of $w$ such that

  $$\forall v \in U \, (v \in \llbracket\varphi\rrbracket \Rightarrow v \in \llbracket\varphi\rrbracket)$$

# Topological Negation

Negation is defined as a shorthand: $\neg\varphi := \varphi \to \bot$

Write $(\mathfrak{X}, x) \models \varphi$ for $x \in [\![\varphi]\!]$.

$(\mathfrak{X}, x) \models \neg\varphi$ iff $x$ has a neighbourhood $U$ such that

$$\forall y \in U\ ((\mathfrak{X}, x) \not\models \varphi)$$

$(\mathfrak{X}, x) \models \neg\neg\varphi$ iff $x$ has a neighbourhood $U$ such that $[\![\varphi]\!]$ is dense in $U$.

Example: $p \vee \neg p$ is not valid topologically.

# Intuitionistic Temporal Logic

**Language**[1] $\mathcal{L}_{\Diamond\Box}$**:** $\varphi, \psi :=$

$$\bot \ \mid \ p \ \mid \ \varphi \wedge \psi \ \mid \ \varphi \vee \psi \ \mid \ \varphi \to \psi \ \mid \ \circ\varphi \ \mid \ \Diamond\varphi \ \mid \ \Box\varphi$$

**Topological** LTL **models:** $(X, S, [\![\cdot]\!])$ where $X$ is a topological space, $S\colon X \to X$ and $[\![\cdot]\!]$ an intuitionistic valuation.

- $[\![\circ\varphi]\!] = S^{-1}[\![\varphi]\!]$?
- $[\![\Diamond\varphi]\!] = \bigcup_{n=0}^{\infty} S^{-n}[\![\varphi]\!]$?
- $[\![\Box\varphi]\!] = \bigcap_{n=0}^{\infty} S^{-n}[\![\varphi]\!]$?

[1] Henceforth all languages have $\circ$, so we only display other tenses.

# Continuity is Important!

The clause

$$[\![\circ\varphi]\!] = S^{-1}[\![\varphi]\!]$$

preserves openness iff $S$ is continuous.

Continuity ensures that

$$[\![\Diamond\varphi]\!] = \bigcup_{n=0}^{\infty} S^{-n}[\![\varphi]\!]$$

also produces open sets.

**Dynamic topological models:** $(X, S, [\![\cdot]\!])$ where $X$ is a topological space, $S\colon X \to X$ is **continuous** and $[\![\cdot]\!]$ an intuitionistic valuation.

# THE CALCULUS $\mathrm{ITL}^0_\Diamond$

ITAUT Intuitionistic propositional axioms

TEMPORAL AXIOMS:

| | |
|---|---|
| $\mathrm{NEXT}_\bot$ | $\neg \circ \bot$ |
| $\mathrm{NEXT}_\wedge$ | $(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi)$ |
| $\mathrm{NEXT}_\vee$ | $\circ(\varphi \vee \psi) \to (\circ\varphi \vee \circ\psi)$ |
| $\mathrm{NEXT}_\to$ | $\circ(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$ |
| $\mathrm{FIX}_\Diamond$ | $(\varphi \vee \circ\Diamond\varphi) \to \Diamond\varphi$ |

RULES:

MP $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

NEC $\dfrac{\varphi}{\circ\varphi}$

MON $\dfrac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$

$\mathrm{IND}_\Diamond$ $\dfrac{\circ\varphi \to \varphi}{\Diamond\varphi \to \varphi}$

# SOUNDNESS OF $\mathrm{ITL}^0_\Diamond$

THEOREM (EXERCISE)
*The calculus* $\mathrm{ITL}^0_\Diamond$ *is sound for the class of dynamical systems.*

$$\textsc{Next}_\leftrightarrow := \circ(\varphi \to \psi) \leftrightarrow (\circ\varphi \to \circ\psi)$$

THEOREM (EXERCISE)
*The calculus* $\mathrm{ITL}^0_\Diamond$ *is sound for the class of dynamical systems with an interior map.*

The clause

$$[\![\Box\varphi]\!] = \bigcap_{n=0}^{\infty} S^{-n}[\![\varphi]\!] \tag{1}$$

does not in general produce open sets, even if $S$ is continuous.

PROPOSITION

*Clause* (1) *yields an open valuation for* $\mathcal{L}_{\Diamond\Box}$ *when* $X$ *is an Alexandroff/poset space.*

**Kremer:**

$$[\![\Box\varphi]\!] = \left(\bigcap_{n=0}^{\infty} S^{-n}[\![\varphi]\!]\right)^{\circ} \tag{2}$$

# KREMER: $\Box p \to \circ\Box p$ FAILS!

**Countermodel:**

$$X = \mathbb{R}$$

$$V(p) = (-\infty, 1)$$

$$S(x) = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

# RESCUING LTL

PROPOSITION

*Over the class of Alexandroff/poset systems, the semantics based on* (1) *validates*

$$\textsc{Fix}_{\Box} \quad \Box\varphi \to (\varphi \wedge \circ\Box\varphi)$$

$$\textsc{Ind}_{\Box} \quad \Box(\varphi \to \circ\varphi) \to (\varphi \to \Box\varphi)$$

PROPOSITION (EXERCISE)

*Over the class of dynamical systems with an interior map, the semantics based on* (2) *also validates the above formulas.*

# GÖDEL-DUMMETT LOGIC

Propositional Gödel-Dummett logic assigns to each formula a value $V \in [0,1]$ so that

- $V(\bot) = 0$
- $V(\varphi \wedge \psi) = \min\{V(\varphi), V(\psi)\}$
- $V(\varphi \vee \psi) = \max\{V(\varphi), V(\psi)\}$
- $V(\varphi \to \psi) = \begin{cases} 1 \text{ if } V(\varphi) \leq V(\psi) \\ V(\psi) \text{ otherwise} \end{cases}$

It is axiomatised over IPC by

GD $(p \to q) \vee (q \to p)$

**Real-valued semantics:** Gödel-Dummett temporal logic GDTL has models $(X, S, V)$ where $S\colon X \to X$ and $V = \{V_x : x \in X\}$ is a family of Gödel-Dummett valuations so that

- $V_x(\circ\varphi) = V_{S(x)}(\varphi)$
- $V_x(\Diamond\varphi) = \sup_{n\in\mathbb{N}} V_{S^n(x)}(\varphi)$
- $V_x(\Box\varphi) = \inf_{n\in\mathbb{N}} V_{S^n(x)}(\varphi)$

# GDTL as an Intuitionistic Logic

**Relational semantics:** Gödel-Dummett logic is also the logic of intuitionistic models which are a **disjoint union of linear orders**.

THEOREM

*A formula of $\mathcal{L}_{\Diamond\Box}$ is valid for its real-valued semantics if and only if it is valid for the class of Alexandroff systems based on a disjoint union of linear orders with an interior map.*

GDTL includes the logic

$$\mathrm{ITL}^0_{\Diamond} + \textsc{gd} + \textsc{Next}_{\leftrightarrow} + \textsc{Fix}_{\Box} + \textsc{Ind}_{\Box}$$

**Open question:** Is this axiomatisation complete?

(Spoiler: Probably not.)

# The Universal Modality

For arbitrary dynamical systems, the universal modality gives a crude (but very useful!) substitute for $\Box$.

**Language $\mathcal{L}_{\Diamond\forall}$:**

$$p \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \circ\varphi \mid \Diamond\varphi \mid \forall\varphi$$

**Semantics for $\forall$:**

$$[\![\forall\varphi]\!] = \begin{cases} X & \text{if } [\![\varphi]\!] = X \\ \varnothing & \text{otherwise} \end{cases}$$

**The universal modality is classical!**

$$\exists\varphi \equiv \neg\forall\neg\varphi$$

# EXPRESSIVITY

**Recall:** A dynamical system $(X, S)$ is **Poincaré recurrent** if whenever $A \subseteq X$ is open and non-empty, there are $x \in A$ and $n > 0$ such that $S^n(x) \in A$.

EXERCISE: This is equivalent to the intuitionistic validity of

$$p \to \neg\neg \circ \Diamond p$$

**Recall:** $(X, S)$ is **minimal** iff for all $x \in X$ and non-empty, open $A \subseteq X$ there is $n > 0$ such that $S^n(x) \in A$.

This is equivalent to the intuitionistic vaility of

$$\exists p \to \Diamond p$$

# THE CALCULUS $\mathrm{ITL}^0_{\Diamond\forall}$

Add the following to $\mathrm{ITL}^0_{\Diamond}$:

| | | | |
|---|---|---|---|
| $\mathrm{K}_\forall$ | $\forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi)$ | $\mathrm{EM}_\forall$ | $\forall\varphi \vee \neg\forall\varphi$ |
| $\mathrm{DIST}_\forall$ | $\forall(\varphi \vee \forall\psi) \to \forall\varphi \vee \forall\psi$ | $\mathrm{T}_\forall$ | $\forall\varphi \to \varphi$ |
| $\mathrm{NEXT}_\forall$ | $\forall\varphi \leftrightarrow \circ\forall\varphi$ | $4_\forall$ | $\forall\varphi \to \forall\forall\varphi$ |
| $\mathrm{NEC}_\forall$ | $\dfrac{\varphi}{\forall\varphi}$ | | |

THEOREM (EXERCISE)
$\mathrm{ITL}^0_{\Diamond\forall}$ *is sound for the class of dynamical systems.*

This logic cannot be Kripke complete due to the formula

$$\forall(\neg p \vee \Diamond p) \to (\Diamond p \vee \neg\Diamond p)$$

# FALSIFYING $\forall(\neg p \vee \Diamond p) \rightarrow (\Diamond p \vee \neg\Diamond p)$

**Countemodel:**

$$X = \mathbb{R}$$

$$V(p) = (1, \infty)$$

$$S(x) = 2x$$

However, it is Kripke-valid.

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