Non-Classical Temporal Logic in Topological Dynamics

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Barcelona, 2024

DAY 2

Before ITL

1997 Artemov, Davoren and Nerode introduced the bi-modal classical logic S4C based on $\circ$ and the **interior semantics** for $\blacksquare$. They proved that

- S4C is Kripke-complete.
- it has the finite model property

2005 Kremer and Mints showed that the above results also hold for S4H, the variant of S4C where f is a **homeomorphism** (equivalently, an interior map).

They introduced Dynamic Topological Logic (DTL), which extends S4C with $\Box$.

They showed it can express **minimality** and **Poincaré recurrence.**

Negative results

2005 Kremer and Mints showed DTL is not complete for Alexandroff spaces.

2006 Konev, Kontchakov, Wolter and Zakaryashev proved that

- DTL is undecidable
- $\mathrm{DTL}_{\mathcal{H}}$, where f is restricted to be a homeomorphism/interior map, is non-axiomatizable

2014 F-D showed that DTL is not finitely axiomatisable.

An Old Hope

2004 Kremer suggested replacing DTL by a version of ITL in an unpublished note.

- Is ITL decidable?
- Is ITL finitely axiomatisable?

Let's find out!

Reminder: Topological ITL

Language $\mathcal{L}_{\Diamond\forall}$**:**

$$\bot \;\mid\; p \;\mid\; \varphi \wedge \psi \;\mid\; \varphi \vee \psi \;\mid\; \varphi \to \psi \;\mid\; \circ\varphi \;\mid\; \Diamond\varphi \;\mid\; \forall\varphi$$

Topological LTL **models:** $(X, S, [\![\cdot]\!])$ where X is a topological space, $S\colon X \to X$ and $[\![\cdot]\!]$ an intuitionistic valuation.

- $[\![\circ\varphi]\!] = S^{-1}[\![\varphi]\!]$
- $[\![\Diamond\varphi]\!] = \bigcup_{n=0}^{\infty} S^{-n}[\![\varphi]\!]$
- $[\![\forall\varphi]\!] = \begin{cases} X & \text{if } [\![\varphi]\!] = X \\ \varnothing & \text{otherwise} \end{cases}$

RESCUING KRIPKE SEMANTICS

ITL is Kripke-incomplete, but many techniques from modal logic are based on these semantics.

Question: Can we still use Kripke semantics to understand ITL over arbitrary spaces?

Answer: Yes we can, as long as we weaken the **functionality** conditions on S.

This idea gives rise to **non-deterministic quasimodels.**

Definition: Type

Fix finite Σ closed under subformulas.

A **type** is a partition $\Phi = (\Phi^{+}, \Phi^{-})$ of Σ satisfying natural **coherence** conditions

$$(p \wedge q, p, q; \Diamond r, r)$$

QUASIMODELS

LABELLED POSET: Triple $(W, \preccurlyeq, \ell)$ where ℓ assigns a type to each $w \in W$ according to the Kripke semantics

WEAK QUASIMODEL: Tuple $(W, \preccurlyeq, R, \ell)$ consisting of a **locally finite** labelled preorder equipped with a **sensible relation:**

- R is **forward-confluent**
- R respects tenses

QUASIMODELS: Weak quasimodels $(W, \preccurlyeq, R, \ell)$ such that

- R is ω**-sensible**
- ℓ is **honest:** Respects $\forall$.

EXAMPLE: Falsify $\forall(\neg p \vee \Diamond p) \to (\Diamond p \vee \neg\Diamond p)$ in a quasimodel.

FROM DYNAMICAL SYSTEMS TO QUASIMODELS

THEOREM

A formula $\varphi \in \mathcal{L}_{\Diamond\forall}$ is valid over the class of dynamical systems iff it is valid over the class of quasimodels

Proof.

(⇒) Define a natural topology and transition function on the set of **realizing paths**

(⇐) Fix a finite set of formulas Σ closed under subformulas

Construct a universal weak quasimodel $\mathbb{M}_\Sigma$

Prove that if φ is falsifiable, then it is falsifiable on some quasimodel $\mathcal{Q} \sqsubseteq \mathbb{M}_\Sigma$

QUASIMODELS BY SIMULATION

A **simulation** E between a weak quasimodel $\mathcal{Q} = (W, \preccurlyeq, R, \ell)$ and a dynamic topological model $\mathcal{M} = (X, S, [\![\cdot]\!])$ is a binary relation

$$E \subseteq W \times X$$

such that

1. E preserves types
2. E is continuous (preimages of opens are open)
3. E is **dynamic:** it is backward confluent for R.

Extracting Quasimodels

Let $\mathcal{Q} = (W, \preccurlyeq, R, \ell)$ be a weak quasimodel, $\mathcal{M} = (X, S, [\![\cdot]\!])$ a dynamic topological model.

Lemma (Exercise)

If $E \subseteq W \times X$ is a dynamic simulation, then the domain of E is a quasimodel.

Our strategy will be to construct a weak quasimodel which surjectively simulates any dynamic topological model.

MOMENTS

We define $\mathbb{M}_\Sigma = (M_\Sigma, \preccurlyeq, R, \ell)$ by

- M_Σ is the set of all **moments:** finite, rooted, tree-like labeled posets
- $\mathfrak{v} \preccurlyeq \mathfrak{w}$ if $\mathfrak{v}$ is an open substructure of $\mathfrak{w}$
- $\mathfrak{v} \; R \; \mathfrak{w}$ if there is a sensible, root-preserving relation between $\mathfrak{v}$ and $\mathfrak{w}$

Fact: $\mathbb{M}_\Sigma$ is a weak quasimodel, but not necessarily a quasimodel.

Strategy for Extracting Quasimodels

The structure $\mathbb{M}_\Sigma = (M_\Sigma, \preccurlyeq, R, \ell)$ amalgamates all finite weak quasimodels but is 'too large':

- It is infinite, despite individual moments being finite.
- It is in general not ω-sensible.

Given a model $\mathfrak{X} = (X, \mathcal{T}, S, [\![\cdot]\!])$, we can identify those elements of M_Σ which **simulate** points in X.

- Given any model with domain X, there is a surjective, dynamic simulation $E^* \subseteq M_\Sigma \times X$.
- The domain of E^* will give us our desired quasimodel.

The Problem with Infinity

The fact that $\mathbb{M}_{\Sigma}$ is infinite has two disadvantages:

- Our proof techniques require moments to be uniformly bounded.
- We can prove that ITL is decidable if every falsifiable formula were falsified in a **finite** quasimodel.

Thus we will identify a **finite** substructure $\mathbb{I}_{\Sigma}$ of $\mathbb{M}_{\Sigma}$ which is still universal.

Irreducible Moments

Denote moments by $\mathfrak{m} = (|\mathfrak{m}|, \preccurlyeq_{\mathfrak{m}}, \ell_{\mathfrak{m}})$.

Definition

- $\mathfrak{w} \sqsubseteq \mathfrak{v}$ if $|\mathfrak{w}| \subseteq |\mathfrak{v}|$, $\preccurlyeq_{\mathfrak{w}} = \preccurlyeq_{\mathfrak{v}} \restriction |\mathfrak{w}|$, and $\ell_{\mathfrak{w}} = \ell_{\mathfrak{v}} \restriction |\mathfrak{w}|$
- $\mathfrak{w} \trianglelefteq \mathfrak{v}$ if $\mathfrak{w} \sqsubseteq \mathfrak{v}$ and there is a continuous, surjective function $\pi \colon |\mathfrak{v}| \to |\mathfrak{w}|$ such that $\ell_{\mathfrak{v}}(v) = \ell_{\mathfrak{w}}(\pi(v))$ for all $v \in |\mathfrak{v}|$ and $\pi^2 = \pi$.
  We say that $\mathfrak{w}$ is a *reduct* of $\mathfrak{v}$ and π is a *reduction*

Definition

A moment $\mathfrak{w}$ is **irreducible** if $\mathfrak{n} \trianglelefteq \mathfrak{m}$ implies $\mathfrak{n} = \mathfrak{m}$.
We denote the set of irreducible moments by I_{Σ} and the restriction of $\mathbb{M}_{\Sigma}$ to I_{Σ} by $\mathbb{I}_{\Sigma}$.

Surjectivity of Simulations

Lemma

If $\mathfrak{m}$ is a moment there is $\mathfrak{n} \trianglelefteq \mathfrak{m}$ which is irreducible and effectively bounded.

Proposition

The relation $\trianglelefteq \subseteq I_\Sigma \times M_\Sigma$ is a surjective, dynamic simulation.

Proposition

Let $\mathfrak{X}$ be any dynamic topological model with domain X and
$E^* \subseteq M_\Sigma \times X$ be the **maximal simulation.**
Let $E_0^* \subseteq I_\Sigma \times X$ be the restriction to I_Σ.
Then, both E_0^* and E^* are surjective, dynamic simulations.

Decidability

Theorem

A formula of $\mathcal{L}_{\Diamond\forall}$ is falsifiable in a dynamic topological model iff it is falsifiable on an effectively bounded quasimodel.

Proof.

Let $E_0^* \subseteq I_\Sigma \times X$ be the maximal simulation, which is a surjective, dynamic simulation. Then, $\mathbb{I}_\Sigma$ restricted to the domain of E_0^* is a finite quasimodel falsifying any formula falsified by $\mathfrak{X}$. $\square$

Corollary

The set of $\mathcal{L}_{\Diamond\forall}$ formulas valid over the class of dynamical systems (with a continuous function) is decidable.

The classical DTL is undecidable for the same class of models.

ALEXANDROFF ITL

Recall: Over Alexandroff/poset models, $\Box$ can be evaluated 'classically'.

THEOREM (BOUDOU ET AL.)

The set of $\mathcal{L}_{\Diamond\Box}$ formulas valid over the class of Alexandroff dynamical systems is decidable.

PROOF SKETCH.

Very similar to the topological case, except that $\mathfrak{m}\ R\ \mathfrak{n}$ is witnessed by a sensible **function**. $\Box$

Logics with Interior Maps

These techniques do not seem to work for logics with interior maps because reductions only preserve forward **or** backward confluence.

Preservation of both seems to require **full** bisimulation.

The decidability of ITL with interior maps is open with $\diamond$ and/or $\square$.

DTL over this class of models is non-axiomatisable.

DECIDABILITY OF GDTL

PROPOSITION
Every Σ-labelled linear moment is bisimilar to one with $\#\Sigma + 1$ elements.

THEOREM
GDTL *is decidable.*

The proof works almost verbatim except that

- moments should be linear
- $\mathfrak{m}\ R\ \mathfrak{n}$ if there is a **fully confluent** sensible relation between them
- the topological unwinding requires a step-by-step method to maintain linearity.

DTL over this class of models is still non-axiomatisable!

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