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Characterising cognitively useful blends: Formalising governing principles of conceptual blending

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ABSTRACT

We propose a model that conceptualises diagrammatic sensemaking and reasoning as blends of image schemas – patterns derived from our perceptual and embodied experiences and interactions with the environment – with the geometric structure of the diagram. Our ultimate goal is to develop an algorithmic method for determining several potential blends that hold cognitive value for observers. Building upon our formal, category-theoretic approach to conceptual blending, we extend it by formalising two governing principles of blending. These principles serve as guides for the blending process, directing the cognitive construction of the blend. As these principles may compete with each other and favour different blend structures, we argue that their combination leads to cognitively useful blends. Through examples of several alternative blends of the geometric configuration of a particular Hasse diagram with the SCALE image schema, we demonstrate the implications of these competing pressures on diagrammatic reasoning. Consequently, this work disambiguates and operationalises the intricacies of conceptual blending, advancing its applicability in computational systems.

1. Introduction

Formal approaches to diagrammatic representation and reasoning often focus on the mapping between the geometric configuration's syntax (i.e., shapes) and the intended concepts they represent (Gurr, 1998; Palmer, 1978; Shimojima, 1996). However, these abstract approaches overlook the active role of the observer in diagrammatic reasoning and assume predefined meanings for each geometric shape. A more cognitively-informed approach to diagrammatic reasoning acknowledges that geometric configurations are not inherently meaningful but rather prompt the observer to structure them into meaningful diagrams by integrating appropriate mental frames (Klein et al., 2006). Scholars suggest that interpreting diagrams involves an active, constructive, and imaginative process on the part of the observer (Cheng et al., 2001; Legg, 2013; May, 1999).

To develop a framework for understanding diagrams that incorporates the embodied nature of our reasoning capacity, we propose modelling diagrammatic inferences as originating from image schemas. Specifically, we envision sensemaking and reasoning with diagrams as networks of conceptual blends, combining suitable image schemas with elements of the diagram's geometry. Image schemas represent recurring patterns of our embodied experiences, acquired through interaction and action in our physical world. These schemas capture the invariant aspects of our experiences since infancy. For instance, by touching and observing different types of containers, we learn that they all possess an *inside* and an *outside*, separated by a *boundary* (Johnson, 1987; Lakoff, 1987). Bourou et al. (2021b) showed how specific image schemas can be blended with the geometric configuration of particular diagrams, resulting in inferences consistent with the intended semantics of the diagrams. They consider a diagram not only the configuration of certain geometric shapes in space; a diagram, as humans make sense of it, is geometry blended with the image schemas that ground the human embodied understanding of the geometry.

Image schemas have previously been used in computational frameworks of conceptual blending to guide the concept creation process (Hedblom et al., 2016; Schorlemmer et al., 2016). In these works, image schemas usually play the role of a common generic structure underlying the mental spaces to be blended (the so-called 'generic space', see Section 2.2). In our approach to diagram sensemaking, however, we follow Bourou et al. (2021b), and image schemas play the role of input spaces to the blend, as they are integrated with the geometric configuration of a diagram. The common generic underlying structure of such a conceptual blend is based on the correspondences found between the geometric configuration of a diagram and the image-schematic structures that support our making sense of it.

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While any schema could potentially be blended with any geometric configuration, not all blends are equally likely to be constructed cognitively by observers to support their inferences and reasoning adequately. Conceptual blending follows governing principles that guide the structuring of blends to possess cognitively desirable features (Fauconnier & Turner, 2002, pp. 325–336), such as being compact and easy to remember or manipulate, while preserving the original structure of the blended concepts. These principles also enable observers to recognise the constituent parts of the blend and understand it. Since the governing principles aim to produce cognitively valuable blends, we believe they can be instrumental in formally characterising suitable blends within our framework for diagram sensemaking.

In this article, we revisit these governing principles and propose a formalisation aligned with the uniform, representation-independent characterisation of conceptual blending introduced by Schorlemmer and Plaza (2021). We argue that our formalisation can serve as a cognitively-inspired heuristic to guide the selection of appropriate image schemas for making sense of a diagram. Furthermore, we demonstrate how these governing principles operate in diagrammatic reasoning through examples of blends between an image schema and the geometric configuration of a diagram. This article's significance lies in its progress towards formally characterising which conceptual blends of image schemas with a given geometric configuration are cognitively preferable and why. By establishing formal criteria for evaluating possible blends, we may algorithmically explore the space of potential meanings of a geometric configuration and gain new insights. Ultimately, our framework aims to bring the theory of conceptual blending from a descriptive realm to a mathematically formal one, enabling its application in computational contexts.

The structure of this article is as follows: Section 2 introduces the key ideas related to this work, including the relevant background on the governing principles and our previous contributions to a mathematical framework for blending. Section 3 restates the formal details of our blending framework and introduces additional definitions that facilitate the description of our newly introduced formalisations of the governing principles later in the same section. Section 4 showcases how the degree of satisfaction of some governing principles can be quantified, analysing four blends of an image schema with a diagram. Finally, Section 5 is a discussion of our contributions, Section 6 reviews related work, and Section 7 includes our conclusions and future work.

2. Background

2.1. Sensemaking and image schemas

Enactivism proposes that cognition and meaning emerge as living organisms (agents) interact with their environment in a goal-directed manner, aiming to grow and sustain themselves (Merleau-Ponty, 1983; Varela, 1991). This process heavily relies on the embodiment of the agent, as the physical body imposes specific requirements for survival, equipped with sensory organs and actuators that shape the agent's perception and interaction within the environment. In line with this notion, *sensemaking* can be understood as the active process of selecting and projecting a structuring frame onto an agent's percepts to construct meaning from them (Klein et al., 2006). A concrete approach to sensemaking involves the utilisation of image schemas and conceptual blending (Fauconnier & Turner, 2002, pp. 104–105). Image schemas represent patterns derived from recurrent bodily experiences acquired early in life, not through the acquisition of propositions, rules, or criteria, but through experiences such as maintaining balance, supporting objects, orienting oneself in space and time, performing body movements, or manipulating objects. Repetitive experiences of a similar nature give rise to the formation of image schemas, which reflect the invariant structure shared among these experiences (Johnson, 1987; Lakoff, 1987).[1]

The primary function of image schemas lies in their ability to shape our perception and experience. It is hypothesised that many concepts we, as humans, utilise – such as time, events, and causes – are structured and comprehended through the unconscious construction of conceptual metaphors grounded in image schemas (Lakoff & Johnson, 1999). Image schemas possess a gestalt-like nature, comprising a set of components arranged in specific relational structures, wherein the meaning of each component arises solely through its relation to all others (Lakoff & Núñez, 2000, p. 31). It is through these properties that observers are capable of integrating image schemas with their experiences, thus making sense of those experiences and extracting meaning from them. To fulfil this function, the structural integrity of image schemas must be preserved during this integration process (Lakoff & Núñez, 2000, p. 42).

2.2. Conceptual blending

In the previous subsection, we briefly discussed sensemaking as the integration of image schemas with our experiences, which can be described through the theory of conceptual blending. To introduce conceptual blending, we must first explore the concept of mental spaces. Mental spaces are "small conceptual packets constructed as we think and talk, for purposes of local understanding and action" (Fauconnier & Turner, 2002, p. 102). They represent coherent and integrated units of information that encompass entities, relations, and properties characterising them. Mental spaces can be constructed based on previously acquired knowledge or current experiences, including exposure to language (Fauconnier, 1994; Fauconnier & Turner, 2002). While they operate in working memory, the construction of mental spaces can also draw upon long-term memory due to their reliance on preexisting knowledge. Elements within one mental space can be corresponded to elements in other mental spaces, enabling cognitive access between them. The central premise of the theory of conceptual blending is that a systematic process of establishing correspondences between different preexisting mental spaces, referred to as 'input spaces', can lead to the emergence of novel meanings. This is particularly evident in the generation of original and unexpected mental spaces, such as the mythical creature Pegasus, which combines the body parts of a bird and a horse. Although less apparent, sensemaking also involves a form of novelty, as stimuli become cognitively structured by agents in new, meaningful ways (Klein et al., 2006). Examples in Fauconnier and Turner (2002) illustrate how conceptual blending can be employed to theorise about such fundamental cognitive processes.

By establishing correspondences between pairs of entities or relations from separate input spaces, a new mental space called a 'blended space' (or simply 'blend') is generated. In this blended space, the elements that are corresponded exhibit novel relationships or are merged with each other. These correspondences are determined by a 'generic space', which represents shared or general aspects to be corresponded between the input spaces. It is not necessary for all elements and relations from the input spaces to be projected into the blend or have correspondences with other input spaces; different subsets can be selectively involved in the blending process. This enables the emergence of a range of alternative structures and meanings. The relations within an individual input space are referred to as 'inner-space relations', while the correspondences between elements of different spaces are termed 'outer-space relations'. The network composed of the input spaces, blended space, generic space, and correspondences among all spaces

[1] In this work, we adopt the convention of denoting specific image schemas, such as CONTAINER, SUPPORT, VERTICALITY, or BALANCE, using uppercase letters.

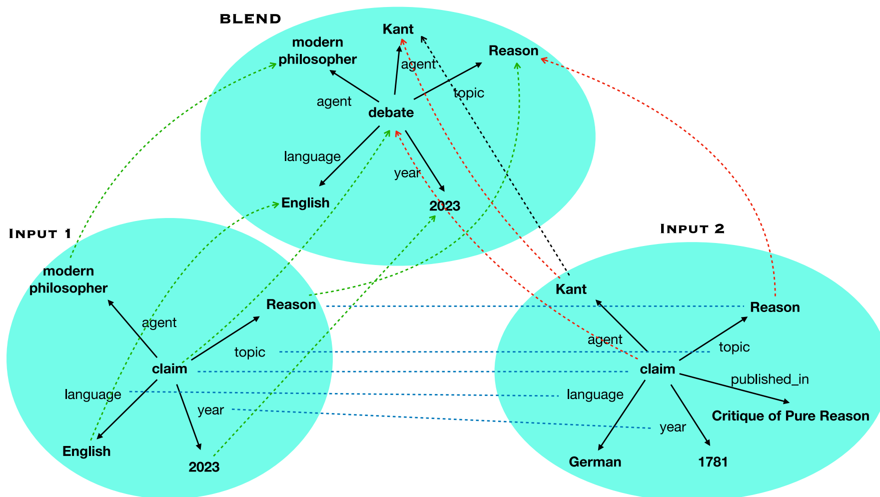


Fig. 1. A visualisation of the conceptual blend corresponding to the debate with Kant. Light blue ellipses represent mental spaces (inputs spaces INPUT 1 and INPUT 2, and blended space BLEND), and dashed lines across them represent outer-space relations: in blue, the cross-space correspondence between input spaces; and in green and red, the selective projections from the input spaces into the blend, respectively. Solid arrows inside the ellipses are inner-space relations. The generic space (not shown) would constitute the mental space capturing the cross-space correspondence: claims done by some agents, in some language, on some topic, in some year.

is known as the ‘integration network’. Meaning is determined by the entire integration network as a whole.

Let us illustrate the above with an example from Fauconnier and Turner (2002, pp. 59–62) which we have modified slightly. The example pertains to a modern philosopher describing how some of his work counters arguments published by Kant in his own work in the following manner:

> I claim that reason is a self-developing capacity. Kant disagrees with me on this point. He says it’s innate, but I answer that that’s begging the question, to which he counters, in Critique of Pure Reason, that only innate ideas have power. But I say to that, What about neuronal group selection? And he gives no answer.

Evidently, the philosopher presents an imaginary in-person debate between himself and Kant. In Fig. 1 we illustrate the conceptual blend capturing this state of affairs. The left oval represents the input space with information about the modern philosopher, and the right one, the input space with information about Kant. The oval at the top denotes the blended space, and dashed lines or arrows between any two spaces denote outer-space relations or projections, respectively. We can see that there are several elements which are present in BLEND which have been projected only from INPUT 1. Some of those are: the language (English) or the year (2023). Some projections from INPUT 2 into BLEND (specifically with the information that Kant’s language was German, and that his claim dates from 1781) would clash with the fact that the modern philosopher is debating in English in 2023. We see how all these outer-space relations and projections allow us to create a complex, detailed, imaginary state of affairs that is nonetheless internally consistent.

2.3. Governing principles

In this section, we introduce the governing principles for good blends identified in Fauconnier and Turner (2002, ch. 16) and aim to provide a more precise and unambiguous description of these principles. Later, in Section 3.1.2, we will formalise some of them. While theoretically any blend can be constructed cognitively, not all blends are useful for reasoning and communication. The governing principles serve as guiding criteria to identify blends that are more likely to be cognitively valuable.

However, there are certain challenges associated with the governing principles, including the lack of experimental support and their inherent ambiguity. Our objective here is to explore whether the governing principles can be effectively used to characterise useful blends within the framework of the uniform model of conceptual blending proposed by Schorlemmer and Plaza (2021), specifically in the domain of diagrammatic sensemaking and reasoning. To achieve this goal, we need to formalise the governing principles in a manner consistent with our mathematical language for describing conceptual blends, namely category theory. By expressing the governing principles using category theory, our formalisation will be generic, making minimal assumptions about the content, form, and formal language used to represent the input spaces. Furthermore, we aim to capture the trade-offs among the principles in our descriptions. Each governing principle favours blends with different structural characteristics, often mutually exclusive. Striking the right balance between these principles is crucial to obtaining cognitively useful blends.

In Fauconnier and Turner (2002), several governing principles are described to some extent, including Compression, Topology, Pattern Completion, Integration, Promoting Vital Relations, Web, Unpacking, and Relevance. In this article, we will formalise two of these principles, namely Integration and Topology, as they meet our criteria of generality and are well-suited for our formal framework.

2.4. Integration and topology

The Integration principle asserts that a blend must consist of an integrated mental space that can be manipulated as a cohesive unit (Fauconnier & Turner, 2002, p. 328). Achieving integration may require selectively projecting only certain parts of the input spaces into the blend to avoid incongruences (which will be discussed shortly). Although the definition of integration in the literature is circular, we can interpret it as follows: the blend should be integrated into a single unit by establishing outer-space relations between elements from different input spaces and projecting them into the blend connected by inner-space relations. Maximum integration is attained when all elements in the blend are linked by an inner-space relation derived from an outer-space relation between the two input spaces. On the other hand, Fauconnier and Turner (2002) also employ the term ‘disintegration’, sometimes interchangeably with ‘incongruence’. This usage can be observed in the example of a philosophical debate involving Kant and a modern philosopher in 1995, conducted in English, which is incongruent with our knowledge of Kant's historical context (living in the 18th century and speaking German) (Fauconnier & Turner, 2002, pp. 125, 329). In general, incongruences and disintegrations arise when there is a clash between elements in terms of some property, making it impossible to merge these conflicting elements in the blend. In all the examples provided by Fauconnier and Turner (2002), this results in integration networks where elements in the blend have outer-space relations with only one of the two input spaces. In Section 3.1.2, we will explore how this understanding of incongruence aligns with our formal definition of the Integration principle.

The Topology principle is satisfied when the topology of inner-space and outer-space relations in the blend mirrors that of the input spaces (Fauconnier & Turner, 2002, p. 327). Specifically, all inner-space relations of the elements in an input space, as well as the outer-space relations among elements from different input spaces, must appear as inner-space relations in the blend. Therefore, the Topology principle exerts a conservative influence, aiming to preserve relations exactly as they were in the input spaces. Consequently, we propose using the term 'Topology Preservation' as a more appropriate name for this principle. To satisfy Topology Preservation, the elements from the two input spaces should either (a) lack any outer-space relations between them, appearing separately but intact in the blend, or (b) have only one-to-one outer-space relations between them, which translate into inner-space relations in the blend, enhancing their integration to some extent while preserving all inner-space relations of the input spaces. Outer-space relations that link multiple elements of one input space to the same element of another would violate Topology Preservation since different inner-space relations from the input spaces would be merged into the same altered relation. Additionally, to fulfil Topology Preservation, all elements from the input spaces should be represented in the blend to ensure that the structure of the input spaces is fully reflected.

3. A mathematical model of blending and its governing principles

When blending mental spaces, incongruences often arise among them, referring to elements from the original spaces that would conflict if present in the blend. An example of such incongruences can be seen in the Kant debate, where two philosophers spoke different languages and lived 200 years apart. Even in the absence of incongruence, elements from one input space that lack correspondence with the other may not be projected into the blend at all. This reflects the cognitive aspect of excluding irrelevant information from a blend tailored to a specific purpose, allowing for more concise and effective reasoning and communication (Fauconnier & Turner, 2002, pp. 313–314, 329, 334).

In the context of the European COINVENT research project (Schorlemmer et al., 2014), Bou et al. (2018) proposed a formal, mathematical framework based on category theory to capture the phenomena of conceptual blending using the concept of 'amalgams'. Originally introduced in Case-Based Reasoning (CBR) to combine cases for problem-solving, amalgams consider all cases, including the combined result, to be partially ordered by a subsumption relation (Ontañón & Plaza, 2010). Subsumption occurs when one case is more general than or equal to another, implying that all information in the latter is contained in the former (Ontañón & Plaza, 2012). Amalgams formalise the process of removing specific information from cases to facilitate their combination without incongruences or when certain information is irrelevant to the problem-solving task at hand. This formalisation involves generalisation operators (see Definition 4), which take one case as input and return another case that subsumes it. This approach allows for computational exploration of the subsumption hierarchy.

Extending this perspective to conceptual blending, we consider the input spaces, generic space, and blend to be part of a subsumption hierarchy capturing how the input spaces can be generalised to eliminate undesired information from the blend. Modelling amalgams in category theory enables us to discuss the principles of conceptual blending independent of specific representation languages used in expressing the inputs. A uniform model of conceptual blending as amalgams using category theory was developed by Schorlemmer and Plaza (2021), combining the implementability of amalgams with the generality of category theory.

Several other computational approaches to conceptual blending have explored the subsumption hierarchy and employed various heuristics to address the exponentially increasing search space, even for simple input spaces (see Section 6 for a more detailed discussion of these approaches). In these implementations, important elements of the input spaces (e.g., axioms, predicates, operations) for a given integration network are predetermined by assigning manual priority indices to each element. These indices are then used as input for various metrics aimed at assessing the structural criteria of integration networks, which effectively reduce the search space for the specific problem at hand. In this article, instead of manually assigning importance indices, which rely on knowledge of the meaning of the input spaces and the intended goals and meaning of the blend, we propose a more general-purpose framework for conceptual blending. This framework remains useful even when applied to domains different from our own or when the meaning of the inputs is unknown. We provide a precise formalisation of some of the governing principles proposed by Fauconnier and Turner (2002). In particular, we aim at formalising two of these principles in a manner that is general-purpose, independent of the specific content of the input spaces, and does not require pre-selection of what is considered important. The conceptual blending framework we adopt is the uniform category-theory framework developed by Schorlemmer and Plaza (2021). To provide technical details relevant to this framework, we present them in Section 3.1.1. Our formal definitions of the governing principles (Section 3.1.2) are guided by this framework.

3.1. The category-theoretic model

We present the uniform model for conceptual blending proposed by Schorlemmer and Plaza (2021) and extend it to formally describe the governing principles. Additionally, we introduce metrics and operators necessary for evaluating these governing principles in the context of specific conceptual blends. The model developed by Schorlemmer and Plaza is rooted in category theory, which provides a general framework for studying mathematical structures by focusing on the structure-preserving mappings between them, rather than the constituents of the structures themselves (Mac Lane, 1998). This emphasis on structure-preserving mappings makes category theory particularly suitable for formalising cognitive operations in a representation-independent manner when specific properties of the entities involved are not assumed (Phillips, 2022).

In category theory, the objects represent the entities of interest, while the structure-preserving mappings (also known as morphisms or arrows) generalise homomorphisms, which are mappings between objects that preserve their internal structure. For conciseness, we omit the introduction of category-theoretic concepts and constructs used in our formalisations in this section. Instead, we refer the reader to Appendix for further details.

Throughout this section, we use a simple graph-based representation of mental spaces as a running example and focus on a specific blend of two mental spaces. In Section 3.1.1, we describe the mathematical model for conceptual blending, while in Section 3.1.2, we reinterpret and formalise selected governing principles within this model.

3.1.1. Conceptual blends as amalgams

Schorlemmer and Plaza (2021) have provided a uniform characterisation of conceptual blending across several representation formalisms by means of a category-theoretic framework. Mental spaces and the outer-space relations between them are taken to be, respectively, the objects and arrows of some suitable category; blending is then modelled by means of amalgams, which are related to the category-theoretic construct of colimit. To illustrate this framework let us take a simple representation of mental spaces by way of graphs.

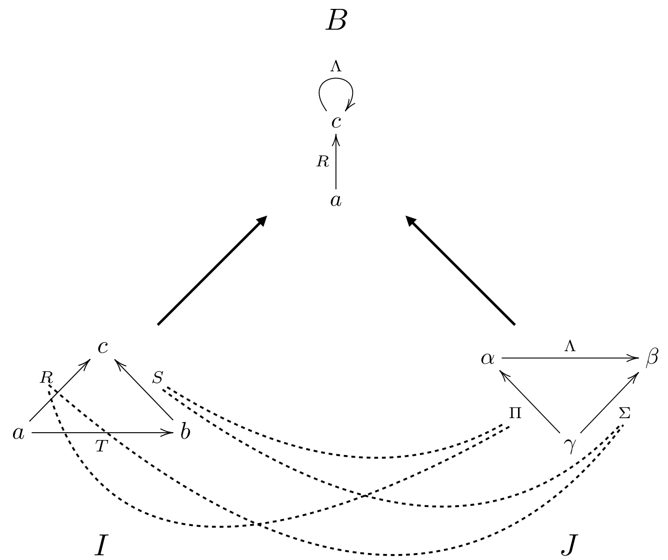
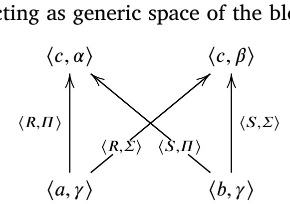


Fig. 2. Blend B of two mental spaces I and J represented by graphs. R, S, and T are binary relations such that *R(a,c)*, *S(b,c)* and *T(a,b)* hold in mental space I (the inner-space relations of I). Π, Σ, and Λ are binary relations such that $\Pi(\gamma,\alpha)$, $\Sigma(\gamma,\beta)$ and $\Lambda(\alpha,\beta)$ hold in mental space J (the inner-space relations of J). The dotted lines show the cross-space relationship underlying the blend (the outer-space relations between I and J): it relates relation R with both relations Π and Σ, and relation S also with both relations Π and Σ. This blend selectively projects only relation $R(a,c)$ from I into the blend while projecting all elements and relations of J. Since, according to the cross-space correspondence, $R(a,c)$ is related to both $\Pi(\gamma,\alpha)$ and $\Sigma(\gamma,\beta)$, the blend B is constituted by the two elements a and c related by R, but also the relation Λ projected from J, relating c with itself (since c is the projection of both α and β).

Example. Let us focus on the particular category **Grph** of graphs and graph homomorphisms. A graph G can act as a simple representation of a mental space, by which the elements of a space are represented as vertices, $V(G)$, and the relations holding between elements are represented as directed edges, $E(G)$. Consequently, only binary relations can be stated with this simple representation.[2]

The idea underlying the framework of Schorlemmer and Plaza (2021) is to look at the selective projection from input spaces I and J to their conceptual blend B (see Section 2.2) in the context of categories of partial arrows (Robinson & Rosolini, 1988). The latter are categories whose arrows are spans $I \leftarrowtail I_0 \longrightarrow B$ consisting of a monic and a total arrow (i.e., *monospans*). Following Schorlemmer and Plaza (2021), we take $\mathsf{Ptl}(\mathscr{C})$ to denote the category whose objects are the same as those of $\mathscr{C}$, and whose arrows are the monospans in $\mathscr{C}$ (i.e., the partial arrows in $\mathscr{C}$).

Example (cont.). Consider the particular blend B of two particular input spaces I and J represented as graphs as shown in Fig. 2 (this is just one of many possible blends). We model this blend in $\mathsf{Ptl}(\mathbf{Grph})$ by representing the cross-space correspondence between input spaces with the span of graph homomorphisms $I \xleftarrow{f} G \xrightarrow{g} J$, where G is the following graph, acting as generic space of the blend:



Vertices are pairs taken from $V(I)\times V(J)$ and edges are pairs taken from $E(I)\times E(J)$; the graph homomorphisms f and g project the first and second component of vertex pairs and edge pairs, respectively. The selective projection from I to B is captured in $\mathsf{Ptl}(\mathbf{Grph})$ by the monospan $I \leftarrowtail I_0 \longrightarrow B$, where I_0 is the subgraph of I consisting only of vertices a and c and the edge R between them.

As in Schorlemmer and Plaza (2021), we shall focus on a subcategory of $\mathsf{Ptl}(\mathscr{C})$ for which the $\mathscr{C}$-monos of the monospans (i.e., of partial arrows in $\mathscr{C}$) are taken only from a distinguished class $\mathscr{M}$ of $\mathscr{C}$-monos, called a ***realm***. We denote this subcategory with $\mathscr{M}-\mathsf{Ptl}(\mathscr{C})$; the class $\mathscr{M}$ can be seen as representing a generalisation space of $\mathscr{C}$-objects:

Definition 1 (*Generalisation of an Object*)**.** Let $\mathscr{C}$ be a category with realm $\mathscr{M}$, and let A be a $\mathscr{C}$-object. We say that $\mathscr{C}$-object A' is a *generalisation* of A if there exists a $\mathscr{C}$-mono $m : A' \rightarrowtail A$ in $\mathscr{M}$.

$\mathscr{C}$-monos into an object A are preordered, and we shall denote this preorder with '$\preccurlyeq_A$': Given $m_1 : A_1 \rightarrowtail A$ and $m_2 : A_2 \rightarrowtail A$, $m_1 \preccurlyeq_A m_2$ iff there exists an arrow $n : A_1 \to A_2$ such that $m_1 = m_2 \circ n$. (Necessarily, n is unique and monic.) In this case, we may also write $A_1 \preccurlyeq_A A_2$.

We now extend the notion of generalisation from objects to spans that represent how two input spaces I and J are put into correspondence through a generic space G:

Definition 2 (*Generalisation of a Span*)**.** Let $\mathscr{C}$ be a category with realm $\mathscr{M}$. Let V be a span $I \xleftarrow{f} G \xrightarrow{g} J$ in $\mathscr{C}$.

- An *outer-space generalisation* of V is a V-diagram (i.e., a span)

  $$I \xleftarrow{f\circ o} G_0 \xrightarrow{g\circ o} J$$

  in $\mathscr{C}$, where $o : G_0 \rightarrowtail G$ is a $\mathscr{C}$-mono in $\mathscr{M}$ representing a generalisation of generic space G. (By generalising G we generalise the cross-space correspondence between I and J.)
- An *inner-space generalisation* of V is a W-diagram

  $$I_0 \longleftarrow f^{-1}(I_0) \rightarrowtail \left(f^{-1}(I_0) \cup g^{-1}(J_0)\right) \leftarrowtail g^{-1}(J_0) \longrightarrow J_0$$

  in $\mathscr{C}$, such that $m : I_0 \rightarrowtail I$ and $n : J_0 \rightarrowtail J$ are $\mathscr{C}$-monos in $\mathscr{M}$ representing generalisations of input spaces I and J, respectively. (By generalising I or J we generalise the internal structure of the input mental spaces.)

In the case of both an outer- and inner-space generalisation, we obtain the W-diagram that is the inner-space generalisation of the span that is an outer-space generalisation of the original span $I \xleftarrow{f} G \xrightarrow{g} J$, namely:

$$I_0 \longleftarrow (f\circ o)^{-1}(I_0) \rightarrowtail \left((f\circ o)^{-1}(I_0) \cup (g\circ o)^{-1}(J_0)\right) \leftarrowtail (g\circ o)^{-1}(J_0) \longrightarrow J_0$$

Notice that the objects of the diagram of a generalisation (be it outer, inner, or both) of a span $I \longleftarrow G \longrightarrow J$ are all subobjects of either G, I or J, and that the diagram is determined by the monic $\mathscr{C}$-arrows $m : I_0 \rightarrowtail I$, $n : J_0 \rightarrowtail J$, and $o : G_0 \rightarrowtail G$. Consequently, we will take this triple $\langle m, n, o\rangle$ of $\mathscr{C}$-monos to refer to a generalisation of a span V. If m and n are the identities, then the generalisation is outer-space; if o is the identity, then the generalisation is inner-space (see Fig. 3). We can extend the ordering on monos componentwise to an ordering of triples of monos, and we will denote this ordering with '$\preccurlyeq_V$'.

We have now all definitions in place to restate Schorlemmer and Plaza's category-theoretic definition of amalgam in the context of this article:

Definition 3 (*Amalgam*)**.** Let $\mathscr{C}$ be a category with realm $\mathscr{M}$. An *amalgam* for a span V in $\mathscr{C}$ is the colimit for a generalisation of V (see Fig. 3).

To generate generalisations of objects and spans for the computation of amalgams we will resort to some generalisation operator by which we can partially explore the generalisation spaces of objects and spans. The number of applications of the generalisation operator (i.e., the

[2] We use the term *graph* as it is usually understood by category theorists, i.e., as a directed multigraph with loops (also called *pseudograph* or *quiver*).

$$
\begin{tikzcd}
 & & B & & \\
I_0 \arrow[urr, "\iota"] \arrow[ddd, tail, "m"'] & & (f\circ o)^{-1}(I_0)\cup(g\circ o)^{-1}(J_0) \arrow[u, "\lambda"'] \arrow[dd, tail, "\overline{m}\cup\overline{n}"] & & J_0 \arrow[ull, "\kappa"'] \arrow[ddd, tail, "n"] \\
 & (f\circ o)^{-1}(I_0) \arrow[ul, "\overline{f\circ o}"] \arrow[ur, tail] \arrow[dr, tail, "\overline{m}"'] & & (g\circ o)^{-1}(J_0) \arrow[ur, "\overline{g\circ o}"'] \arrow[ul, tail] \arrow[dl, tail, "\overline{n}"] & \\
 & & G_0 \arrow[dd, tail, "o"] & & \\
I & & & & J \\
 & & G \arrow[ull, "f"] \arrow[urr, "g"'] & &
\end{tikzcd}
$$

Fig. 3. Amalgam with apex B. The $\mathscr{C}$-arrows ι, κ and λ are the injections of the colimit; the $\mathscr{C}$-monos m, n, and o represent the generalisation of the span; arrows $\overline{m}$ and $\overline{n}$ are the pullbacks of m and n along $f\circ o$ and $g\circ o$, respectively; and $\overline{m}\cup\overline{n}$ is the inclusion mono of the union $(f\circ o)^{-1}(I_0)\cup(g\circ o)^{-1}(J_0)$ in G_0.

generalisation steps) needed for some generalisation determines a measure of generalisation that further underlies the similarity value of amalgams.

Definition 4 (*Generalisation Operator*). Let $\mathscr{C}$ be a category with realm $\mathscr{M}$. A ***generalisation operator*** γ for $\mathscr{C}$ is a set-valued function such that, given a $\mathscr{C}$-object A, for every $A' \in \gamma(A)$ there exists a $\mathscr{C}$-mono $m : A' \rightarrowtail A$ in $\mathscr{M}$. Given a set of $\mathscr{C}$-objects $\mathscr{A}$, we write $\gamma(\mathscr{A})$ for the set $\bigcup_{A\in\mathscr{A}} \gamma(A)$; and $\gamma^l(A)$ stands for $\underbrace{\gamma(\gamma(\cdots\gamma(A)))}_{l \text{ times}}$l.

For any $\mathscr{C}$-object A, a generalisation operator γ is said to be *locally finite* if $\gamma(A)$ is a finite set; it is said to be *proper* if $A \notin \gamma(A)$; and it is said to be *complete* if, for any $\mathscr{C}$-mono $m : A' \rightarrowtail A$ in $\mathscr{M}$, there exists a $l \geq 0$, such that $A' \in \gamma^l(A)$. A generalisation operator that is locally finite, proper, and complete is said to be *ideal*.

Definition 5 (*Reachable Generalisation of an Object*). Let $\mathscr{C}$ be a category with realm $\mathscr{M}$. Let γ be a generalisation operator for $\mathscr{C}$. Let $m : A' \rightarrowtail A$ be a $\mathscr{C}$-mono in $\mathscr{M}$. If there exists $l \geq 0$ such that A' is isomorphic to a $\mathscr{C}$-object in $\gamma^l(A)$, we say that A' is a γ-*reachable* generalisation of A (or simply a *reachable* generalisation, when γ is clear from the context). We take the minimum such l as the *measure* of this generalisation and denote it with $\lambda(m)$.

Example (cont.). A generalisation operator for **Grph** could be the one that, given a graph, yields the set of all its subgraphs obtained by removing either one single edge or else one single vertex, provided that this vertex has no incoming nor outgoing edges (note that this operator is ideal). Consequently, for graph I as shown in Fig. 2, the graph I_0 consisting of the edge $a \xrightarrow{R} c$ is a subgraph of I, and thus a reachable generalisation of I, obtained by removing edges S and T, and vertex b. The measure of this generalisation captured by the **Grph**-mono $m : I_0 \rightarrowtail I$ is $\lambda(m) = 3$.

Given two generalisations A_1 and A_2 of A, if A_1 is a reachable generalisation of A_2, then $A_1 \preccurlyeq_A A_2$. (The converse only holds if the generalisation operator is ideal.) We say that a generalisation operator γ *is coherent*, if the measures of γ-reachable generalisations are anti-monotonic with respect to $\preccurlyeq_A$, i.e., for all generalisations A_1 and A_2 of A (with $\mathscr{C}$-monos $m_1 : A_1 \rightarrowtail A$ and $m_2 : A_2 \rightarrowtail A$ in $\mathscr{M}$), $m_1 \preccurlyeq_A m_2$ implies $\lambda(m_1) \geq \lambda(m_2)$.

The measure λ is only one of many possible ways to define a measure of generalisation. We could, for instance, imagine a generalisation operator defined by way of several generalisation rules that have different weights attached to them, so that the measure of generalisation does not only depend on the number of generalisation steps l, but also on the weight of each step. With weighted measures, one would be able to capture the salience of the structural elements that are generalised or the cognitive diversity of individuals that do the conceptual blending. This might be relevant, for example, when taking into account cultural differences of conceptual blending in melody generation (Kaliakatsos-Papakostas & Cambouropoulos, 2019). For explanatory reasons, and without loss of generality, we have opted in this article to stay with the simple measure as defined in Definition 5.

Given two reachable generalisations of the same object, together with their measures of generalisation, we can now define their similarity, relative to the object they are generalisations of:

Definition 6 (*Similarity of Generalisations*). Let A' and A'' be two reachable generalisations of A. Let $m : A^* \rightarrowtail A$ be the reachable generalisation of A with maximum $\lambda(m)$, of which both A' and A'' are reachable generalisations; consequently, there exist $\mathscr{C}$-monos $m' : A' \rightarrowtail A^*$ and $m'' : A'' \rightarrowtail A^*$ in $\mathscr{M}$:

$$
\begin{tikzcd}
A' \arrow[dr, tail, "m'"'] & & A'' \arrow[dl, tail, "m''"] \\
 & A^* \arrow[d, tail, "m"'] & \\
 & A &
\end{tikzcd}
$$

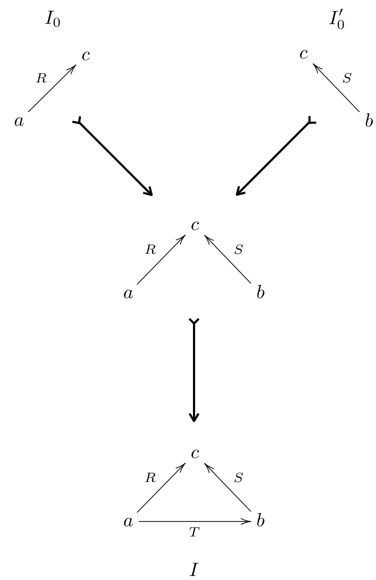
The *similarity* of A' and A'' with respect to A is given by:

$$\sigma(A', A'', A) = \frac{1 + \lambda(m)}{1 + (\lambda(m) + \lambda(m') + \lambda(m''))}$$

We can see that $\sigma(A', A'', A) = 1$ when A' and A'' are isomorphic, because then $A^* \cong A' \cong A''$, and in this case the numerator and the denominator of the fraction are equal. In contrast, we get values for $\sigma(A', A'', A)$ closer to 0 when $A^* \cong A$, because then A' and A'' are generalisations with no common structure.

Instead of $\sigma(A', A'', A)$, we will also denote this similarity as a two-argument function of the respective $\mathscr{C}$-monos, i.e., $\sigma(m \circ m', m \circ m'')$.

Example 3.1.1. Let I_0' be a subgraph of I consisting of vertices b and c and edge S. The similarity of I_0 and I_0' with respect to I is $\sigma(I_0, I_0', I) = \frac{1+1}{1+(1+2+2)} = \frac{2}{6}$, as shown here:



The similarity of generalisations can be extended from objects to spans by defining a measure of generalisation for the triples of monos underlying these generalisations, for instance by taking $\lambda(\langle m, n, o\rangle) = \lambda(m)+\lambda(n)+\lambda(o)$. This forms also the basis for a similarity of amalgams of some span V, provided they can be computed as colimits for reachable generalisations of V:

Definition 7 (*Similarity of Amalgams*)**.** Let V be a span, and let A_1 and A_2 be two amalgams for V, such that A_1 and A_2 are the apexes of the colimits for reachable generalisations V_1 and V_2 of V, respectively. The *similarity of* A_1 *and* A_2 is given by the similarity of V_1 and V_2.

3.1.2. Formalising governing principles

In this subsection, we present our formalisation of the governing principles Integration and Topology Preservation. This formalisation reflects our understanding of these principles, as described in Section 2.3, and is in line with the category-theoretic framework of conceptual blending summarised in Section 3.1.1.

Integration. The Integration principle demands that a blend incorporates the structure of the input spaces into a single whole. This is achieved by associating the elements and relations of each input space with outer-space relations, which become inner-space relations in the blend. Therefore, maximum integration would be reached when all elements in the blend are associated with some inner-space relation that emerges from an outer-space relation between the input spaces. In our formal approach, the only such relation we are considering is identity (identifying elements and relations from the given input spaces by way of a cross-space correspondence).

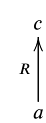
The degree of integration of a blend is thus given by how well it incorporates the input spaces into a whole; i.e., whether each component constituting the internal structure of the blended space integrates components of *both* input spaces, or else the blended space includes some disintegrated fragment, one that is the projection of the elements or relations of only one of the two input spaces. We will thus define the degree of integration of a blend by the similarity it has to an alternative blend of the same input spaces (and assuming the same cross-space correspondence) that integrates *all* the structure that is selectively projected from *both* input spaces.

Example (cont.). In our example (Fig. 2), graph B representing a possible blended space of I and J has vertices a, c and edge R that indeed integrate vertices and edges of both graphs I and J; but edge Λ only comes from J, and not from I. In **Grph**, maximal integration is achieved when the vertex maps and edge maps constituting the graph homomorphisms into the blend are surjective. Lifting this into our category-theoretic framework for blends by means of amalgams, we assert that the amalgam B' that best integrates input spaces I and J is the one whose arrows of the amalgam's underlying colimit for a generalisation of span $I \leftarrow G \rightarrow J$ are all epic. Furthermore, we focus on the smallest such generalisation, i.e., the one obtained by the smallest number of applications of the generalisation operator, so as to keep most of the structure of input spaces.

Definition 8 (*Degree of Integration*)**.** Let B be a conceptual blend, modelled as the apex of an amalgam for span V in $\mathscr{C}$, such that B is the colimit apex for a γ-reachable generalisation $\langle m, n, o\rangle$ of V. The degree of B's compliance with the Integration principle, $\mathsf{Int}(B)$, is given by the maximum similarity measure between generalisation $\langle m, n, o\rangle$ and a minimal γ-reachable generalisation $\langle m', n', o'\rangle$ of V for which there is a colimit whose arrows are all epic (let us call such a colimit an *epic colimit*):

$$\begin{aligned}
\mathsf{Int}(B) &= \max_{\langle m', n', o'\rangle \in \mathscr{W}} \sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle) \\
\mathscr{W} &= \operatorname*{arg\,min}_{\langle m', n', o'\rangle \in \mathscr{V}} \lambda(\langle m', n', o'\rangle) \\
\mathscr{V} &= \{\langle m', n', o'\rangle \mid \langle m', n', o'\rangle \text{ is a } \gamma\text{-reachable generalisation} \\
&\quad \text{of } V \text{ with epic colimit}\}
\end{aligned}$$

Example (cont.). In our example (Fig. 2), a minimal γ-reachable generalisation of $I \leftarrow G \rightarrow J$ with epic colimit (i.e., a colimit whose arrows are all epic) is the span $I_0' \leftarrow G \rightarrow J_0'$ where I_0' is the graph obtained by removing edge T from I (one generalisation step), and J_0' is the graph obtained by removing edge Λ from J (also one generalisation step). The colimit (pushout) of this span is (isomorphic to) the following graph B':



B' has indeed epic arrows both from I_0' and J_0'. B's degree of integration is given by the similarity of B's underlying generalisation of span $I \leftarrow G \rightarrow J$ (which is $I_0 \leftarrow G \rightarrow J$) with B''s underlying generalisation (which is $I_0' \leftarrow G \rightarrow J_0'$). A γ-reachable generalisation of $I \leftarrow G \rightarrow J$ with maximal measure of which both $I_0 \leftarrow G \rightarrow J$ and $I_0' \leftarrow G \rightarrow J_0'$ are generalisations of is the span $I_0' \leftarrow G \rightarrow J$. Consequently, $\mathsf{Int}(B) = \frac{2}{5}$.

Topology preservation. The Topology Preservation principle demands that the inner-space relations between elements of an input space, as well as the outer-space relations among elements of different input spaces, appear as inner-space relations in the blend. This way, relations are faithfully reflected in the blend as they were stated in the input spaces. In our formalisation, outer-space relations become identities in the blend. Therefore, preserving the topology means that each element of I and J has outer-space relations with exactly one element of the other, resulting in every element of I and J having an outer-space relation with exactly one element in the blend.

The degree of topology preservation of a blend is thus given by how well it retains the original structure of the input spaces in the whole; i.e., whether the components constituting the internal structure of the input spaces that are projected to the blend space are kept separate, or else the blended space fuses some of these components, thereby not retaining their distinctiveness. We will thus define the degree of topology preservation of a blend by the similarity it has to an alternative blend of the same input spaces (and assuming the same cross-space correspondence), such that *all* the structure of that is selectively projected into this alternative blend is preserved.

Example (cont.). In our example (Fig. 2), graph B representing a possible blended space of I and J indeed retains the structure $R(a,c)$ of input space I; nonetheless, the distinct elements α and β of J are fused into element c of the blend, and so the relations Π and Σ are fused into R. In **Grph**, maximal topology preservation is achieved when the vertex maps and edge maps constituting the graph homomorphisms into the blend are injective. Lifting this, as done with our formalisation of the Integration principle, into our category-theoretic framework for blends by means of amalgams, we assert that the amalgam B' that best preserves the topology of input spaces I and J is the one whose arrows of the amalgam's underlying colimit for a generalisation of span $I \leftarrow G \rightarrow J$ are all monic. Moreover, as with the Integration principle, we focus on the least general generalisation of the span so as to keep most of the structure of input spaces.

Definition 9 (*Degree of Topology Preservation*)**.** Let B be a conceptual blend, modelled as the apex of an amalgam for span V in $\mathscr{C}$, such that B is the colimit apex for a γ-reachable generalisation $\langle m,n,o\rangle$ of V. The degree of B's compliance with the Topology Preservation principle, $\mathsf{Top}(B)$, is given by the maximum similarity measure between generalisation $\langle m,n,o\rangle$ and a minimal generalisation $\langle m',n',o'\rangle$ of V for which there is a colimit whose arrows are all monic (let us call such a colimit a *monic colimit*):

$$\begin{aligned}
\mathsf{Top}(B) &= \max_{\langle m',n',o'\rangle \in \mathscr{W}} \sigma(\langle m,n,o\rangle, \langle m',n',o'\rangle)\\
\mathscr{W} &= \operatorname*{arg\,min}_{\langle m',n',o'\rangle \in \mathscr{V}} \lambda(\langle m',n',o'\rangle)\\
\mathscr{V} &= \{\langle m',n',o'\rangle \mid \langle m',n',o'\rangle \text{ is a } \gamma\text{-reachable generalisation}\\
&\qquad \text{of } V \text{ with monic colimit}\}
\end{aligned}$$

Example (cont.). In our example (Fig. 2), a minimal γ-reachable generalisation of $I \leftarrow G \rightarrow J$ with monic colimit is the span $I \leftarrow G_0 \rightarrow J$ where G_0 is the graph obtained by removing from G the edges $\langle R,\Pi\rangle$, $\langle R,\Sigma\rangle$, and $\langle S,\Sigma\rangle$ and the vertices $\langle a,\gamma\rangle$ and $\langle c,\beta\rangle$ (five generalisation steps). The colimit (pushout) of this span is (isomorphic to) the following graph B':

$$\begin{tikzcd}
 & c \arrow[r, "\Lambda"] & \beta \\
a \arrow[ur, "R"] \arrow[r, "T"'] & b \arrow[u, "S"'] \arrow[ur, "\Sigma"'] &
\end{tikzcd}$$

B' has indeed monic arrows from I and J. B's degree of topology preservation is given by the similarity of B's underlying generalisation of span $I \leftarrow G \rightarrow J$ (which is span $I_0 \leftarrow G \rightarrow J$) to B''s underlying generalisation (which is $I \leftarrow G_0 \rightarrow J$). For this particular example, a generalisation of $I \leftarrow G \rightarrow J$ with maximal measure of which both $I_0 \leftarrow G \rightarrow J$ and $I \leftarrow G_0 \rightarrow J$ are generalisations of, is actually the identity $\langle id_I, id_G, id_J\rangle$, and $\lambda(\langle id_I, id_G, id_J\rangle) = 0$. Consequently, $\mathsf{Top}(B) = \frac{1}{8}$.

Notice that, in our example using graphs, a blend whose underlying colimit is epic (and thus displays maximal integration) is obtained by inner-space generalisation (i.e., generalisation of the input spaces), while a blend whose underlying colimit is monic (and thus displays maximal topology preservation) is obtained by outer-space generalisation (i.e., generalisation of the generic space). This feature is not only particular to graphs, but to any mathematical structure that can be characterised as a *presheaf*. This is relevant for our case study of diagrammatic reasoning in Section 4, below, and we state it by way of the following theorems:

Theorem 1. *Let* $\mathbf{Psh}(\mathscr{C})$ *be the category of presheaves on* $\mathscr{C}$. *Let* V *be a span* $I \xleftarrow{f} G \xrightarrow{g} J$ *in* $\mathbf{Psh}(\mathscr{C})$. *A minimal generalisation of* V *(i.e., a maximal triple of* $\mathbf{Psh}(\mathscr{C})$*-monos* $\langle m,n,o\rangle$ *with respect to* $\preccurlyeq_V$*) for which the colimit is epic, is given by way of the epi-mono factorisations of* $f = m \circ \bar{f}$ *and* $g = n \circ \bar{g}$ *through their respective images* $m : Im(f) \rightarrowtail I$ *and* $n : Im(g) \rightarrowtail J$*, and taking* $o = id_G$.

Proof. Since $\mathbf{Psh}(\mathscr{C})$ is regular, every arrow can be factored as the composition of an epi followed by a mono, and this factorisation is unique (up to isomorphism). Let $f = m \circ \bar{f}$ and $g = n \circ \bar{g}$ be the unique epi-mono factorisations of the arrows of span V. The generalisation of V given by $\langle m,n,id_G\rangle$ is thus $Im(f) \overset{\bar{f}}{\twoheadleftarrow} G \overset{\bar{g}}{\twoheadrightarrow} Im(g)$, and its colimit (here a pushout) is epic because pushouts preserve epis. Let us now assume there exist a less general generalisation $\langle m',n',id_G\rangle$ of V for which the colimit is also epic, with $m' : I' \rightarrowtail I$, $m \preccurlyeq_I m'$, and $n' : J' \rightarrowtail J$, $n \preccurlyeq_J n'$ (id_G is already maximal with respect to $\preccurlyeq_G$). This generalisation of V is thus $I' \xleftarrow{\bar{f}'} G \xrightarrow{\bar{g}'} J'$, where $\bar{f}'$ is the pullback of f along m', and $\bar{g}'$ is the pullback of g along n'. Since we have assumed that the colimit for this generalisation is epic (here again a pushout), we have, by Lemma 1 (see below), that $\bar{f}'$ and $\bar{g}'$ are epis. But since epi-mono factorisations are unique (up to isomorphism), we have that $I' \cong Im(f)$ and $J' \cong Im(g)$. $\square$

The proof of Theorem 1 above uses the following lemma stating the relationship of epis with pushouts in categories of presheaves.

Lemma 1. *Let* $\mathbf{Psh}(\mathscr{C})$ *be the category of presheaves on* $\mathscr{C}$. *Let* V *be a span* $I \xleftarrow{f} G \xrightarrow{g} J$ *in* $\mathbf{Psh}(\mathscr{C})$. *The pushout of* V *formed by arrows* $\iota : I \rightarrow B$ *and* $\kappa : J \rightarrow B$ *always exists, and if* ι *(resp.* κ*) is epic, then* g *(resp.* f*) is epic.*

Proof. It is easy to check that the lemma holds for the particular case of $\mathbf{Psh}(\mathscr{C}) = \mathbf{Set}$ (i.e., when $\mathscr{C}$ is the category with a unique object and its identity arrow): if ι (resp. κ) is surjective, so is g (resp. f). $\mathbf{Psh}(\mathscr{C})$ is cocomplete (and hence has pushouts) because it is the functor category $\mathbf{Func}(\mathscr{C}^{op}, \mathbf{Set})$ and $\mathbf{Set}$ is cocomplete. Being a functor category, its colimits are constructed pointwise. Since epis can be characterised as pushouts, epis are also constructed pointwise; and because the lemma holds for $\mathbf{Set}$, we have that it also holds for $\mathbf{Psh}(\mathscr{C})$. $\square$

Theorem 2. *Let* $\mathbf{Psh}(\mathscr{C})$ *be the category of presheaves on* $\mathscr{C}$. *Let* V *be a span* $I \xleftarrow{f} G \xrightarrow{g} J$ *in* $\mathbf{Psh}(\mathscr{C})$. *A minimal generalisation of* V *(i.e., a maximal triple of* $\mathbf{Psh}(\mathscr{C})$*-monos* $\langle m,n,o\rangle$ *with respect to* $\preccurlyeq_V$*) for which the colimit is monic, is given by way of a maximal* $\mathbf{Psh}(\mathscr{C})$*-mono* $o : G_0 \rightarrowtail G$ *with respect to* $\preccurlyeq_G$ *such that* $f \circ o$ *and* $g \circ o$ *are both monic, and taking* $m = id_I$ *and* $n = id_J$.

Proof. The generalisation of V given by $\langle id_I, id_J, o\rangle$ is the span $I \xleftarrow{f \circ o} G_0 \xrightarrow{g \circ o} J$ of monos. Since categories of presheaves are regular, and pushouts preserve regular monos, we have that the pushout (colimit) for this generalisation is monic. Let us suppose, however, that there exists a less general generalisation of V given by the triple of monos $\langle id_I, id_J, o'\rangle$ (i.e., $o : G_0 \rightarrowtail G$ factors through $o' : G_0' \rightarrowtail G$) for which the colimit formed by arrows $\iota : I \rightarrowtail B$, $\kappa : J \rightarrowtail B$, and $\lambda : G_0' \rightarrowtail B$ is also monic. Since $\lambda = \iota \circ f \circ o' = \kappa \circ g \circ o'$, we have that $f \circ o'$ and $g \circ o'$ are both monic as well. And, since o is maximal with respect to $\preccurlyeq_G$ such that $f \circ o$ and $g \circ o$ are both monic, we have that o' factors through o, and thus G_0' and G_0 are isomorphic. $\square$

4. Governing principles in diagram sensemaking

Making sense of a diagram is a cognitive process that involves multiple blends with various image schemas (Bourou et al., 2021c). However, in this section, we shift our focus to highlight the governing principles underlying conceptual blending. To stay focused and not get caught within the complexity of the integration network describing the sensemaking of every meaning-carrying aspect of a diagram, we shall examine conceptual blends involving only *one* image schema and the geometric configuration of only *one* particular diagram, namely a Hasse diagram, modelling alternative ways of making sense of a single aspect of it, namely the levels that the Hasse diagram presents. By investigating these blends, we aim to demonstrate the practicality of the

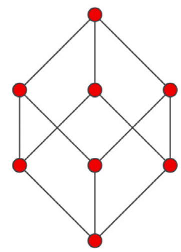


Fig. 4. A Hasse diagram representing a poset of eight elements.

formalised governing principles in understanding how we make sense of certain diagrammatic structure and engage in reasoning processes related to it. We argue that this case study, although constrained, still provides insights into the application of the governing principles as defined in a representation-independent manner to guide our understanding and reasoning with diagrams in general.

Recall that a Hasse diagram represents a partially ordered set (poset). It consists of points and lines, with each point representing one element of the poset. Assuming elements x, y and z of a poset ordered by the '<' relation, then x is shown in a lower position than y, and connected by a line with it, if and only if $x < y$ and there is no element z such that $x < z$ and $z < y$. Some Hasse diagrams represent ranked posets, i.e., posets for which all maximal chains have the same finite length. This means that the elements of the poset can be organised into levels, corresponding to elements with the same rank, i.e., the same number of steps away from some minimal element (see Fig. 4).

4.1. A category of specifications for modelling diagrams as conceptual blends

The formalisation of conceptual blending and its governing principles given in Section 3 is generic and representation-independent. This is its strength, as it does not need to commit to any particular representation language for mental spaces and their relations, mappings, and projections. However, in order to compute specific blends and their degrees of compliance with respect to the governing principles, we need concrete specifications of their input spaces with both their inner- and outer-space relations, expressed in a particular representation language. That is, we need to commit to a particular category that models spaces and their structure-preserving mappings. The generalisation operator underlying our mathematical model of conceptual blending and its governing principles also needs to be defined with respect to the chosen category.

Our computational approach to model the sensemaking of diagrams uses theory presentations in many-sorted first-order logic with equality ($\mathrm{FOL}^=$), to specify the logical structure of both image schemas and geometric configurations of diagrams. We implement these theory presentations using *basic specifications* expressed in the Common Algebraic Specification Language (CASL), which has $\mathrm{FOL}^=$ as one of its sublanguages (Astesiano et al., 2002). We further use the Heterogeneous Tool Set (HETS) to perform computations over these specifications (Mossakowski et al., 2007).

A *specification* $S = (\Sigma, \Gamma)$ consists of a declaration of its *signature* Σ (i.e., its sorts, operations – which comprises constants and function symbols – and predicates) and a finite set Γ of *axioms* written with the signature symbols of Σ. Given a category of $\mathrm{FOL}^=$ signatures, if $\sigma : \Sigma \to \Sigma'$ is one of its *signature morphisms*, and $S = (\Sigma, \Gamma)$ and $S' = (\Sigma', \Gamma')$ are two specifications, then σ can also be seen as a *specification morphism* $\sigma : S \to S'$ whenever $\Gamma' \vDash \sigma(\Gamma)$; i.e., whenever all $\mathrm{FOL}^=$ models of the axioms in Γ' are also models of the translation along σ of the axioms in Γ. Specifications and specification morphisms as given above constitute a category, **Spec**. Let **Spec*** be the subcategory of **Spec** whose specification morphisms are *axiom preserving*, i.e., when $\sigma(\Gamma) \subseteq \Gamma'$. In this section, we take category **Spec*** for modelling particular diagrams of image schemas with geometric configurations by way of amalgams in **Spec***.

A straightforward generalisation operator for **Spec*** is the one that, given a specification $S = (\Sigma, \Gamma)$, yields specifications S_i obtained by:

- either removing one of its axioms in Γ, thus yielding $S_i = (\Sigma, \Gamma_i)$, where $1 \leq i \leq n$, with n denoting the number of axioms in Γ (note that the identity signature morphism on Σ is an axiom-preserving specification morphism from S_i to S);
- or else removing one of the signature symbols in Σ, provided it does not occur in Γ, thus yielding $S_i = (\Sigma_i, \Gamma)$, where $1 \leq i \leq m$, with m denoting the number of signature symbols in Σ not occurring in Γ (note that the inclusion signature morphism $\Sigma_i \subseteq \Sigma$ is an axiom-preserving specification morphism from S_i to S);

As for the specification of the geometric configurations of diagrams, we have based their signatures on Qualitative Spatial Reasoning (QSR) formalisms as proposed by Egenhofer and Herring (1994) and Hernández (1991), which allow us to describe qualitatively certain aspects of the topology and geometry of the diagram, e.g., if the shapes occurring in a diagram are of sort *Point* or *Line*, and which are their topological relations or relative positions.[3]

4.2. Case study: Making sense of levels in a hasse diagram

In this subsection, we analyse four alternative ways of modelling the sensemaking of different levels in the Hasse diagram of Fig. 4. We have argued that such sensemaking can be understood as a conceptual blend of the geometric configuration constituting the diagram with some image schema capturing our embodied understanding of the levels sensed in the diagram. In this particular case, our sensemaking would be driven by the SCALE image schema, which structures both the quantitative and qualitative aspects of our experience – such as when we group objects or when we add them to a pile; or when we experience one light brighter as the other, or one pain as more intense than another – organising these aspects in different grades that account for our experience of more, the same, and less quantity or quality (Johnson, 1987, p. 121–124).

Furthermore, we have proposed to model conceptual blends as amalgams in an appropriate category serving as representation formalism for the mental spaces participating in the blends, and we have chosen the category **Spec*** of basic CASL specifications and axiom-preserving CASL signature morphisms for that aim.

Let us now assume that the particular geometric configuration of the Hasse diagram we shall be analysing is specified in CASL as in Specification 1. Let us also assume that the SCALE image schema is specified in CASL as in Specification 2, which includes the generic specification SCALE of the SCALE schema, as well an extension of it, 4SCALE, specifying a particular four-grade scale.[4]

4.2.1. Intended blending

The first blend we analyse captures what we may assume to be our intended perception of the Hasse diagram of Fig. 4 as organised in four levels, with elements depicted as laying at the same horizontal in the

[3] The complete CASL specifications are available at https://saco.csic.es/index.php/s/iKWyj8Kn3nQGKMf. The sensemaking of more diagrams is modelled in Bourou et al. (2021c) and the Hasse diagram discussed here is also discussed in detail in Bourou et al. (2021b). There it is also explained how the geometry can be formalised using our choice of QSR formalisms.

[4] Note that we distinguish the image schema as such (indicated with sans-serif uppercase letters) from the CASL specification of the schema (indicated with small caps).

Specification 1 The geometric configuration of the Hasse diagram of Fig. 4 by way of its minimal signature and independent facts. It uses predicates that state the relative position of points with respect to each other as proposed by Hernández (1991) (assuming the perspective of an observer in front of the diagram as drawn on a sheet of paper lying on a desktop); and predicates that state the topological relationship of points and lines as proposed by Egenhofer and Herring (1994) (We show only a fragment; the complete specification is available at https://saco.csic.es/index.php/s/iKWyj8Kn3nQGKMf.)

spec HASSE =
 sorts *Point, Line*
 ops $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8 : Point$
 $l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, l_{10}, l_{11}, l_{12} : Line$
 preds *leftPP, backPP, leftBackPP, rightBackPP* : *Point* × *Point*
 intersectLL : *Line* × *Line*
 intersectLP : *Line* × *Point*

 %% Axioms for HASSE:
 %% relative position of pairs of points:
 • *rightBackPP*(p_1, p_2)
 • *backPP*(p_1, p_2)
 • *leftBackPP*(p_1, p_4)
 • *rightBackPP*(p_1, p_5)
 • *leftBackPP*(p_1, p_7)
 ⋮
 %% topological relation – here, intersection – of lines with points:
 • *intersectLP*(l_1, p_1) • *intersectLP*(l_1, p_2)
 • *intersectLP*(l_2, p_1) • *intersectLP*(l_2, p_3)
 ⋮
 %% topological relation – here, intersection – of lines with lines:
 • *intersectLL*(l_5, l_6)
 • *intersectLL*(l_7, l_8)
end

diagram, sharing the same level. This is captured by a cross-space correspondence that relates certain points of the geometric configuration with certain grades of the SCALE image schema.

Therefore, the sort *Point* of specification HASSE (see Specification 1) is put into correspondence with the sort *Grade* of specification 4SCALE (see Specification 2). Moreover, point p_1 is put into correspondence with grade g_4; points p_2, p_3, and p_4 with grade g_3; points p_5, p_6, and p_7 with grade g_2; and point p_8 with grade g_1. Predicates *backPP*, *leftBackPP*, and *rightBackPP* of HASSE are put into correspondence with the predicate *more* of 4SCALE since this is how we understand points situated 'farther back' in the diagram (using the terminology proposed by Hernández (1991)) as being 'more' in the scale of grades. Specification 3 shows this cross-space correspondence represented as a span V of CASL signature morphisms (called 'views' in CASL) from a common generic space.

The conceptual blend capturing this sensemaking of the Hasse diagram as being structured in four levels (Fig. 4) would thus be modelled by an amalgam over this span. There are many possible amalgams over the same span, since an amalgam is ultimately the colimit for *some* generalisation of a given span, and there are many possible such generalisations. Let us now see which, of all potential amalgams, would be the appropriate one to model this intended sensemaking of the Hasse diagram as organised in four levels.

Notice that, if we were to take the colimit for the span given in Specification 3 without generalising it, we would get a blend that is inconsistent, assuming the intended meaning of the relative positioning predicates we have taken from Hernández (1991) for our specification of the Hasse diagram as in Specification 1. That is so because the predicate *leftPP* is irreflexive, and this fact clashes with the statements *leftPP*(g_2, g_2) and *leftPP*(g_3, g_3) that we would get in the colimit for the given span.

Thus, the actual blend that we think is most faithful to our intended sensemaking of the Hasse diagram as structured in four levels, is the one that ignores and does not project the *leftPP* relationship between points into the blend. This blend would be modelled by an amalgam, whose underlying colimit is determined by a generalisation V_0 of the original span V, where we generalise the specification HASSE to HASSE_0 by removing all occurrences of the *leftPP* predicate. The result would be the intended blend HASSE_4SCALE_BLEND as in Specification 4.

Let us now examine the degrees of integration and topology preservation of this blend according to our governing principles presented in Section 3.1.2.

Integration. For measuring the integration of the blend, we need to identify a minimal generalisation V_e of our original span V for which the colimit is epic. Since **Spec** signatures (and hence also **Spec*** signatures) are families of sets, they can be characterised as presheaves. Consequently, we have that such minimal generalisation is given by way of epi-mono factorisations (see Theorem 1), which amounts to making a minimal inner-space generalisation of HASSE and 4SCALE so that *each* entity of the colimit integrates entities projected from *both* input spaces. (This generalisation removes the structure that is written in black in Specification 4.) Specification 5 shows these generalised input spaces.

The degree of integration of HASSE_4SCALE_BLEND is then

$$\mathrm{Int}(\textsc{hasse\_4scale\_blend}) = \sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$$

where m : HASSE_0 $\rightarrowtail$ HASSE is the **Spec***-mono of the inclusion of HASSE_0 into HASSE, and n and o are identity arrows; and where m' : HASSE_E $\rightarrowtail$ HASSE and n' : 4SCALE_E $\rightarrowtail$ 4SCALE are the **Spec***-monos of the inclusions of HASSE_E and 4SCALE_E into HASSE and 4SCALE, respectively, and o' is again the identity arrow (since the generic space is not generalised).

The similarity $\sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$ of span generalisations is calculated in terms of the generalisation steps of a given generalisation operator (see Definition 6). Assuming an operator that removes one axiom at a time, or that removes one signature symbol at a time (provided it does not occur in any axiom), and taking into account that HASSE_E is a generalisation of HASSE_0, (i.e. there exists **Spec***-mono m'' : HASSE_E $\rightarrowtail$ HASSE_0 such that $m' = m \circ m''$), we have that:

$$\begin{tikzcd}
V_0 \arrow[dr, tail, "id"'] & & V_e \arrow[dl, tail, "\langle m'', n', o'\rangle"] \\
& V_0 \arrow[d, tail, "\langle m,n,o\rangle"'] & \\
& V &
\end{tikzcd}$$

And thus:

$$\begin{aligned}
&\sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle) \\
&\quad = \frac{1 + \lambda(\langle m, n, o\rangle)}{1 + \lambda(\langle m, n, o\rangle) + \lambda(id) + \lambda(\langle m'', n', o'\rangle)} \\
&\quad = \frac{1 + (\lambda(m) + \lambda(n) + \lambda(o))}{1 + (\lambda(m) + \lambda(n) + \lambda(o)) + 0 + (\lambda(m'') + \lambda(n') + \lambda(o))} \\
&\quad = \frac{1 + (5 + 0 + 0)}{1 + (5 + 0 + 0) + 0 + (41 + 14 + 0)} \\
&\quad = \frac{6}{61} \approx 0.0984
\end{aligned}$$

Topology. For measuring the topology preservation of the blend, we need to identify a minimal general generalisation V_m of our original span V for which the colimit is monic. Since we have that **Spec** signatures are presheaves, this amounts to making a minimal outer-space generalisation of the cross-space correspondence between input spaces (by generalising the specification GENERIC of the span modelling the cross-space correspondence), such that we obtain a one-to-one correspondence between input spaces, so that entities that are separate in one input space, stay separate and are not fused in the blend. Specification 6 shows this generalisation.

Specification 2 Generic SCALE image schema and its extension specifying a particular four-grade scale.

spec SCALE =
  sorts *ScaleSchema, Scale, Grade*
  ops $scale : ScaleSchema \to Scale$
  preds $inScale : Grade \times Scale$
    $more, less : Grade \times Grade$

  %% Axioms for SCALE:
  $\forall s : ScaleSchema;\ c : Scale;\ x, y, z : Grade$
  • $\exists ! sc : ScaleSchema \bullet scale(sc) = c$
  • $(inScale(x, scale(s)) \wedge inScale(y, scale(s)) \wedge \neg(x = y)) \Rightarrow (more(x, y) \vee more(y, x))$
  • $more(x, y) \Rightarrow \exists ! sc{:}ScaleSchema \bullet inScale(x, scale(sc)) \wedge inScale(y, scale(sc))$
  • $less(x, y) \Leftrightarrow more(y, x)$

  • $\neg more(x, x)$ **%irreflexive**
  • $(more(x, y) \wedge more(y, z)) \Rightarrow more(x, z)$ **%transitive**
  • $more(x, y) \Rightarrow \neg more(y, x)$ **%antisymmetric**
end

spec 4SCALE = SCALE
then ops $s : ScaleSchema$
  $g_1, g_2, g_3, g_4 : Grade$

  • $inScale(g_1, scale(s))$
  • $inScale(g_2, scale(s))$
  • $inScale(g_3, scale(s))$
  • $inScale(g_4, scale(s))$

  • $more(g_4, g_3)$
  • $more(g_3, g_2)$
  • $more(g_2, g_1)$
end

Specification 3 Cross-space correspondence between the Hasse geometric configuration and the four-grade SCALE schema, specified by means of a span of CASL signature morphisms (called 'views') from a common generic space—a specification constituted only of signature elements and no axioms.

spec GENERIC =
  sorts S
  ops $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 : S$
  preds $r_1, r_2, r_3 : S \times S$
end

view $h4s_1$: GENERIC **to** HASSE =
  $S \mapsto Point,$
  $c_1 \mapsto p_1,\ c_2 \mapsto p_2,\ c_3 \mapsto p_3,\ c_4 \mapsto p_4,\ c_5 \mapsto p_5,\ c_6 \mapsto p_6,\ c_7 \mapsto p_7,\ c_8 \mapsto p_8,$
  $r_1 \mapsto backPP,\ r_2 \mapsto leftBackPP,\ r_3 \mapsto rightBackPP$
end

view $h4s_2$: GENERIC **to** 4SCALE =
  $S \mapsto Grade,$
  $c_1 \mapsto g_4,\ c_2 \mapsto g_3,\ c_3 \mapsto g_3,\ c_4 \mapsto g_3,\ c_5 \mapsto g_2,\ c_6 \mapsto g_2,\ c_7 \mapsto g_2,\ c_8 \mapsto g_1,$
  $r_1 \mapsto more,\ r_2 \mapsto more,\ r_3 \mapsto more$
end

The degree of topology preservation of HASSE_4SCALE_BLEND is then

$$\mathrm{Top}(\text{HASSE\_4SCALE\_BLEND}) = \sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$$

where m : HASSE_0 $\rightarrowtail$ HASSE is the **Spec***-mono of the inclusion of HASSE_0 into HASSE, and n and o are identity arrows; and where now m' and n' are identity arrows (since input spaces are not generalised), while o' : GENERIC_M $\rightarrowtail$ GENERIC is the **Spec***-mono of the inclusion of GENERIC_M into GENERIC.

Assuming the same generalisation operator as above, and taking into account that the most general generalisation of V of which both V_0 and V_m are generalisations, is V itself, we have that:

$$\begin{tikzcd} V_0 \arrow[dr, tail, "\langle m,n,o\rangle"'] & & V_m \arrow[dl, tail, "\langle m',n',o'\rangle"] \\ & V \arrow[d, tail, "id"'] & \\ & V & \end{tikzcd}$$

And thus:

$$\begin{aligned}
&\sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)\\
&\quad = \frac{1 + \lambda(id)}{1 + \lambda(id) + \lambda(\langle m, n, o\rangle) + \lambda(\langle m', n', o'\rangle)}\\
&\quad = \frac{1 + 0}{1 + 0 + (\lambda(m) + \lambda(n) + \lambda(o)) + (\lambda(m') + \lambda(n') + \lambda(o'))}\\
&\quad = \frac{1 + 0}{1 + 0 + (5 + 0 + 0) + (0 + 0 + 6)}\\
&\quad = \frac{1}{12} \approx 0.0833
\end{aligned}$$

4.2.2. Nongeneralised blending

The second blend we analyse is based on the same cross-space correspondence as specified with the span of Specification 3, but modelled by the amalgam whose underlying colimit is the one for the nongeneralised span. Recall from our discussion above that the resulting blend turns out to be an inconsistent specification. In what follows we show what the integration and topology-preservation measures of this blend are. Let us name this blend NONGEN_HASSE_4SCALE_BLEND.

Integration. For measuring the integration of the blend, we compute the similarity of the generalisation V_e as in Section 4.2.1 above (for which the colimit is epic) with the nongeneralised span V:

$$\mathrm{Int}(\text{NONGEN\_HASSE\_4SCALE\_BLEND}) = \sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$$

Specification 4 Conceptual blend modelling our sensemaking of the Hasse diagram of Fig. 4 with its points organised in four levels. Highlighted in teal are the parts of the blend that integrate structure projected from *both* input spaces; the parts in black are projections only coming from either one or the other input space. We have renamed the integrated structure to highlight that the blend now specifies how entities that can be understood to be both points and grades (*Point-Grade*) are related via a relation (*back-more*) that states simultaneously the ‘back’ and ‘more’ relation we read off the Hasse diagram as we make sense of it through our embodied cognition.

spec HASSE_4SCALE_BLEND =
  sorts *Line, ScaleSchema, Scale, Point-Grade,*
  ops pg_1, pg_2, pg_3, pg_4 : *Point-Grade*
    $l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, l_{10}, l_{11}, l_{12}$: *Line*
    scale : *ScaleSchema* → *Scale*
    s : *ScaleSchema*
  preds *back-more*, *less* : *Point-Grade* × *Point-Grade*
    intersectLL : *Line* × *Line*
    intersectLP : *Line* × *Point-Grade*
    inScale : *Point-Grade* × *Scale*

  %% Axioms for HASSE_4SCALE_BLEND:
  • $back\text{-}more(pg_4, pg_3)$
  • $back\text{-}more(pg_4, pg_2)$
  • $back\text{-}more(pg_3, pg_2)$
  • $back\text{-}more(pg_3, pg_1)$
  • $back\text{-}more(pg_2, pg_1)$

  %% topological relation – here, intersection – of lines with point-grades:
  • $intersectLP(l_1, pg_4)$ • $intersectLP(l_1, pg_3)$
  • $intersectLP(l_2, pg_4)$ • $intersectLP(l_2, pg_3)$
  • $intersectLP(l_3, pg_4)$ • $intersectLP(l_3, pg_3)$
  ⋮

  • $inScale(pg_1, scale(s))$
  • $inScale(pg_2, scale(s))$
  • $inScale(pg_3, scale(s))$
  • $inScale(pg_4, scale(s))$

  $\forall s : ScaleSchema;\ c : Scale;\ x, y, z : Grade$
  • $\exists! sc : ScaleSchema \bullet scale(sc) = c$
  • $(inScale(x, scale(s)) \wedge inScale(y, scale(s)) \wedge \neg(x = y)) \Rightarrow (back\text{-}more(x, y) \vee back\text{-}more(y, x))$
  • $back\text{-}more(x, y) \Rightarrow \exists! sc{:}ScaleSchema \bullet inScale(x, scale(sc)) \wedge inScale(y, scale(sc))$
  • $less(x, y) \Leftrightarrow back\text{-}more(y, x)$

  • $\neg back\text{-}more(x, x)$ **%irreflexive**
  • $(back\text{-}more(x, y) \wedge back\text{-}more(y, z)) \Rightarrow back\text{-}more(x, z)$ **%transitive**
  • $(back\text{-}more(x, y) \Rightarrow \neg back\text{-}more(y, x)$ **%antisymmetric**
end

where now m, n, and o are identity, and $m' : $ HASSE_E $\rightarrowtail$ HASSE and n' : 4SCALE_E $\rightarrowtail$ 4SCALE are the **Spec***-monos of the inclusions of HASSE_E and 4SCALE_E into HASSE and 4SCALE, respectively, and o is again the identity arrow, as we had for Section 4.2.1.

As before, we compute the similarity $\sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$ of span generalisations in terms of the generalisation operator used in Section 4.2.1. Consequently, we have that:

$$\begin{tikzcd} V \arrow[dr, "id"'] & & V_e \arrow[dl, "\langle m', n', o'\rangle"] \\ & V \arrow[d, "id"] & \\ & V & \end{tikzcd}$$

And thus:

$$\begin{aligned}\sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle) &= \frac{1 + \lambda(id)}{1 + \lambda(id) + \lambda(id) + \lambda(\langle m', n', o'\rangle)} \\ &= \frac{1 + 0}{1 + 0 + 0 + (\lambda(m') + \lambda(n') + \lambda(o))} \\ &= \frac{1}{1 + (46 + 14 + 0)} \\ &= \frac{1}{61} \approx 0.0164\end{aligned}$$

Topology. For measuring the topology preservation of the blend, we compute the similarity of the generalisation V_m as in Section 4.2.1 above (for which the colimit is monic) with the nongeneralised span V:

$$\mathsf{Top}(\text{NONGEN\_HASSE\_4SCALE\_BLEND}) = \sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$$

where m, n, o, m', and n' are identity arrows, and $o' : $ GENERIC_M $\rightarrowtail$ GENERIC is the **Spec***-mono of the inclusion of GENERIC_M into GENERIC as in Section 4.2.1. Consequently, we have that:

$$\begin{tikzcd} V \arrow[dr, "id"'] & & V_m \arrow[dl, "\langle m', n', o'\rangle"] \\ & V \arrow[d, "id"] & \\ & V & \end{tikzcd}$$

Specification 5 Inner-space generalisation of the geometric configuration of the Hasse diagram of Fig. 4 (fragment) in which only the information about points and their relative position is kept (except for *leftPP*); and inner-space generalisation of the four-grade scale schema in which only the information about the four grades and their ordering with *more* is kept.

spec HASSE_E =
  sorts *Point*
  ops $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$: *Point*
  preds *backPP, leftBackPP, rightBackPP* : *Point* × *Point*

  %% Axioms forHASSE_E:
  %% relative position of pairs of points:
  • $rightBackPP(p_1, p_2)$
  • $backPP(p_1, p_2)$
  • $leftBackPP(p_1, p_4)$
  • $rightBackPP(p_1, p_5)$
  • $leftBackPP(p_1, p_7)$
  ⋮
end

spec 4SCALE_E =
  sorts *Grade*
  ops g_1, g_2, g_3, g_4 : *Grade*
  preds *more* : *Grade* × *Grade*

  %% Axioms for 4SCALE_E:
  • $more(g_4, g_3)$
  • $more(g_3, g_2)$
  • $more(g_2, g_1)$

  $\forall x, y, z$: *Grade*
  • $\neg more(x, x)$ **%irreflexive**
  • $(more(x, y) \wedge more(y, z)) \Rightarrow more(x, z)$ **%transitive**
  • $more(x, y) \Rightarrow \neg more(y, x)$ **%antisymmetric**
end

Specification 6 Outer-space generalisation of the cross-space correspondence (the span of generic space and signature morphisms) so as to get a one-to-one correspondence of input-space structure.

spec GENERIC_M =
  sorts *S*
  ops c_1, c_3, c_6, c_8 : *S*
  preds r_1 : *S* × *S*
end

view $h4s_{m1}$: GENERIC_M **to** HASSE =
  S ↦*Point*,
  $c_1 \mapsto p_1$, $c_3 \mapsto p_3$, $c_6 \mapsto p_6$, $c_8 \mapsto p_8$,
  $r_1 \mapsto backPP$
end

view $h4s_{m2}$: GENERIC_M **to** 4SCALE =
  S ↦*Grade*,
  $c_1 \mapsto g_4$, $c_3 \mapsto g_3$, $c_6 \mapsto g_2$, $c_8 \mapsto g_1$,
  $r_1 \mapsto more$
end

And thus:

$$\begin{aligned}\sigma(\langle m,n,o\rangle,\langle m',n',o'\rangle) &= \frac{1+\lambda(id)}{1+\lambda(id)+\lambda(id)+\lambda(\langle m',n',o'\rangle)}\\ &= \frac{1+0}{1+0+0+(\lambda(m')+\lambda(n')+\lambda(o))}\\ &= \frac{1}{1+(0+0+6)}\\ &= \frac{1}{7} \approx 0.1429\end{aligned}$$

4.2.3. Only-ranked blending

The third blend we analyse is based on the same cross-space correspondence as specified with the span of Specification 3 but modelled by the amalgam whose underlying colimit is the span in which we have generalised the Hasse geometry keeping only the points and its relative positioning. This geometry is the one given in HASSE_E (see Specification 5). As in Section 4.2.1, this is an asymmetric blend (only one of the input spaces is generalised) and it captures the case of focusing only on how the points in the diagram are ranked, disregarding completely how they are connected with lines. Let us name this blend ONLY-RANKED_HASSE_4SCALE_BLEND.

Integration. For measuring the integration of the blend, we compute the similarity of the generalisation V_e as in Section 4.2.1 above (for which the colimit is epic) with the generalisation V_1 of span V for which HASSE is generalised to HASSE_E:

$$\mathrm{Int}(\text{ONLY-RANKED\_HASSE\_4SCALE\_BLEND}) = \sigma(\langle m,n,o\rangle,\langle m',n',o'\rangle)$$

where now m : HASSE_E ↣ HASSE and n' : 4SCALE_E ↣ 4SCALE are the **Spec***-monos of the inclusions of HASSE_E and 4SCALE_E into HASSE and 4SCALE, respectively, and n, o, m', and o' are identity arrows.

As before, we compute the similarity $\sigma(\langle m,n,o\rangle,\langle m',n',o'\rangle)$ of span generalisations in terms of the generalisation operator used in Section 4.2.1. Consequently, we have that:

$$\begin{tikzcd} V_1 \arrow[dr, tail, "id"'] & & V_e \arrow[dl, tail, "\langle m',n',o'\rangle"] \\ & V_1 \arrow[d, tail, "\langle m,n,o\rangle"'] & \\ & V & \end{tikzcd}$$

And thus:

$$\begin{aligned}&\sigma(\langle m,n,o\rangle,\langle m',n',o'\rangle)\\ &= \frac{1+\lambda(\langle m,n,o\rangle)}{1+\langle m,n,o\rangle+\lambda(id)+\lambda(\langle m',n',o'\rangle)}\\ &= \frac{1+(\lambda(m)+\lambda(n)+\lambda(o))}{1+(\lambda(m)+\lambda(n)+\lambda(o))+0+(\lambda(m')+\lambda(n')+\lambda(o'))}\\ &= \frac{1+(46+0+0)}{1+(46+0+0)+(0+14+0)}\\ &= \frac{47}{61} \approx 0.7705\end{aligned}$$

Topology. For measuring the topology preservation of the blend, we compute the similarity of the generalisation V_m as in Section 4.2.1 above (for which the colimit is monic) with the span V_1:

$$\mathrm{Top}(\text{ONLY-RANKED\_HASSE\_4SCALE\_BLEND}) = \sigma(\langle m,n,o\rangle,\langle m',n',o'\rangle)$$

where m : HASSE_E ↣ HASSE and o' : GENERIC_M ↣ GENERIC are the **Spec***-monos of the inclusions of HASSE_E and GENERIC_M into HASSE and GENERIC, respectively; and n, o, m', and n' are identity arrows. Consequently, we have that:

$$\begin{tikzcd} V_1 \arrow[dr, tail, "\langle m,n,o\rangle"'] & & V_m \arrow[dl, tail, "\langle m',n',o'\rangle"] \\ & V \arrow[d, tail, "id"'] & \\ & V & \end{tikzcd}$$

And thus:

$$\sigma(\langle m,n,o\rangle,\langle m',n',o'\rangle)$$

$$
\begin{aligned}
&= \frac{1 + \lambda(id)}{1 + \lambda(id) + \lambda(\langle m, n, o \rangle) + \lambda(\langle m', n', o' \rangle)} \\
&= \frac{1 + 0}{1 + 0 + (\lambda(m) + \lambda(n) + \lambda(o)) + (\lambda(m') + \lambda(n') + \lambda(o'))} \\
&= \frac{1}{1 + (46 + 0 + 0) + (0 + 0 + 6)} \\
&= \frac{1}{53} \approx 0.0189
\end{aligned}
$$

4.2.4. Disjoint blending

Let us analyse now a limit case, namely the extreme case of total non-integration, that is, when the blend is just a disjoint union of the HASSE and 4SCALE specifications. This blend is modelled with the amalgam whose underlying colimit is for the span obtained by generalising GENERIC to the empty specification. Let us name this blend DISJOINT_HASSE_4SCALE_BLEND.

Integration. For measuring the integration of the blend, we compute the similarity of the generalisation V_e as in Section 4.2.1 above (for which the colimit is epic) with the span $V_\emptyset$ that we obtain by removing all entities from GENERIC, with the same generalisation operator as in Section 4.2.1.

$$\mathrm{Int}(\text{DISJOINT\_HASSE\_4SCALE\_BLEND}) = \sigma(\langle m, n, o \rangle, \langle m', n', o' \rangle)$$

where m are n are identities, and o : EMPTY $\rightarrowtail$ GENERIC is the **Spec***-mono of the inclusions of EMPTY into GENERIC; and m' and n' are the **Spec***-monos of the inclusions of HASSE_E and 4SCALE_E into HASSE and 4SCALE, respectively, and o' is the identity arrow, as we had for Section 4.2.1. Consequently, we have that:

$$
\begin{tikzcd}
V_\emptyset \arrow[dr, tail, "\langle m,n,o \rangle"'] & & V_e \arrow[dl, tail, "\langle m',n',o' \rangle"] \\
& V \arrow[d, tail, "id"'] & \\
& V &
\end{tikzcd}
$$

And thus:

$$
\begin{aligned}
&\sigma(\langle m, n, o \rangle, \langle m', n', o' \rangle) \\
&= \frac{1 + \lambda(id)}{1 + \lambda(id) + \lambda(\langle m, n, o \rangle) + \lambda(\langle m', n', o' \rangle)} \\
&= \frac{1 + 0}{1 + 0 + (\lambda(m) + \lambda(n) + \lambda(o)) + (\lambda(m') + \lambda(n') + \lambda(o'))} \\
&= \frac{1 + 0}{1 + 0 + (0 + 0 + 12) + (46 + 14 + 0)} \\
&= \frac{1}{73} \approx 0.01370
\end{aligned}
$$

Topology. For measuring the topology preservation of the blend, we compute the similarity of the generalisation V_m as in Section 4.2.1 above (for which the colimit is monic) with the span $V_\emptyset$:

$$\mathrm{Top}(\text{DISJOINT\_HASSE\_4SCALE\_BLEND}) = \sigma(\langle m, n, o \rangle, \langle m', n', o' \rangle)$$

where m are n are identities, and o : EMPTY $\rightarrowtail$ GENERIC is the **Spec***-mono of the inclusions of EMPTY into GENERIC; and m' and n' are also identities, and o' is the **Spec***-mono of the inclusions of GENERIC_M into GENERIC, as in Section 4.2.1. Taking into account that EMPTY is a generalisation of GENERIC_M, (i.e., there exists **Spec***-mono o'' : EMPTY $\rightarrowtail$ GENERIC_M such that $o = o' \circ o''$), we have that:

$$
\begin{tikzcd}
V_\emptyset \arrow[dr, tail, "\langle m,n,o'' \rangle"'] & & V_m \arrow[dl, tail, "id"] \\
& V_m \arrow[d, tail, "\langle m',n',o' \rangle"'] & \\
& V &
\end{tikzcd}
$$

Specification 7 SCALE schema of three grades, specified as an extension of the SCALE specification.

spec 3SCALE = SCALE
then ops s : *ScaleSchema*
g_1, g_2, g_3 : *Grade*

- $inScale(g_1, scale(s))$
- $inScale(g_2, scale(s))$
- $inScale(g_3, scale(s))$

- $more(g_3, g_2)$
- $more(g_2, g_1)$

end

Specification 8 Cross-space correspondence between the Hasse geometric configuration and the three-grade SCALE schema to capture a vertically-layered Hasse diagram, specified by means of a span of CASL signature morphisms (called 'views') from the common space GENERIC (see Specification 3).

view $h3s_1$: GENERIC **to** HASSE =
$S \mapsto$ *Point*,
$c_1 \mapsto p_1$, $c_2 \mapsto p_2$, $c_3 \mapsto p_3$, $c_4 \mapsto p_4$, $c_5 \mapsto p_5$, $c_6 \mapsto p_6$, $c_7 \mapsto p_7$, $c_8 \mapsto p_8$,
$r_1 \mapsto$ *leftPP*, $r_2 \mapsto$ *leftBackPP*, $r_3 \mapsto$ *rightBackPP*
end

view $h3s_2$: GENERIC **to** 3SCALE =
$S \mapsto$ *Grade*,
$c_2 \mapsto g_3$, $c_5 \mapsto g_3$, $c_1 \mapsto g_2$, $c_3 \mapsto g_2$, $c_6 \mapsto g_2$, $c_8 \mapsto g_2$, $c_4 \mapsto g_1$, $c_7 \mapsto g_1$,
$r_1 \mapsto$ *more*, $r_2 \mapsto$ *more*, $r_2 \mapsto$ *less*
end

And thus:

$$
\begin{aligned}
&\sigma(\langle m, n, o \rangle, \langle m', n', o' \rangle) \\
&= \frac{1 + \lambda(\langle m', n', o' \rangle)}{1 + \lambda(\langle m', n', o' \rangle) + \lambda(\langle m, n, o'' \rangle) + \lambda(id)} \\
&= \frac{1 + (\lambda(m') + \lambda(n') + \lambda(o'))}{1 + (\lambda(m') + \lambda(n') + \lambda(o')) + (\lambda(m) + \lambda(n) + \lambda(o'') + \lambda(id))} \\
&= \frac{1 + (0 + 0 + 6)}{1 + (0 + 0 + 6) + (0 + 0 + 6) + 0} \\
&= \frac{7}{13} \approx 0.5385
\end{aligned}
$$

4.2.5. Vertically-levelled blending

The three blends above are amalgams for the same cross-space correspondence as given by the span of Specification 3 relating horizontally positioned points in a Hasse diagram with four grades of SCALE schema. Let us now model how we would make sense of the same Hasse diagram if we were to group the points into "vertical levels", as it were, that increase horizontally from right to left. This requires a different correspondence of points with grades, and thus an extension of SCALE specifying a three-grade scale, as shown in Specification 7. The cross-space correspondence is then the span shown in Specification 8.

Analogous to the blend analysed in Section 4.2.1, a blend that captures the sensemaking of the Hasse diagram as organised in three vertical levels, is the one that ignores and does not project the *backPP* relation between points into the blend. This blend would be modelled by an amalgam, whose underlying colimit is determined by a generalisation V_0 of the original span V, where we generalise the specification HASSE to HASSE_0† by removing all occurrences of the *backPP* predicate. The resulting blend would be HASSE_3SCALE_BLEND as shown in Specification 9.

Specification 9 Conceptual blend that captures the sensemaking of the Hasse diagram of Fig. 4 with its points grouped in three vertical levels, as it were. Highlighted in teal are the parts of the blend that fuse entities projected from *both* input spaces; the parts in black are projections only coming from either one or the other input space. We have renamed the fused entities to highlight that the blend now specifies how entities that can be understood to be both points and grades (point-grades) are related via relations (left-more, right-less) that state simultaneously the 'left' and 'more' relations (or the 'right' and 'less' relations) we read off the Hasse diagram.

spec HASSE_3SCALE_BLEND =
  sorts *Point-Grade, Line, ScaleSchema, Scale*
  ops pg_1, pg_2, pg_3 : *Point-Grade*
    $l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, l_{10}, l_{11}, l_{12}$: *Line*
    scale : *ScaleSchema* → *Scale*
    s : *ScaleSchema*
  preds *left-more, right-less* : *Point-Grade* × *Point-Grade*
    intersectLL : *Line* × *Line*
    intersectLP : *Line* × *Point-Grade*
    inScale : *Point-Grade* × *Scale*

  %% Axioms for HASSE_3SCALE_BLEND:
  • *left-more*(pg_2, pg_1) • *right-less*(pg_1, pg_2)
  • *left-more*(pg_3, pg_2) • *right-less*(pg_2, pg_3)
  • *left-more*(pg_3, pg_1) • *right-less*(pg_1, pg_3)

  %% topological relation—here, intersection—of lines with point-grades:
  • *intersectLP*(l_1, pg_2) • *intersectLP*(l_1, pg_3)
  • *intersectLP*(l_2, pg_2)
  • *intersectLP*(l_3, pg_2) • *intersectLP*(l_3, pg_1)
  ⋮

  • *inScale*$(pg_1$,*scale*$(s))$
  • *inScale*$(pg_2$,*scale*$(s))$
  • *inScale*$(pg_3$,*scale*$(s))$

  ∀t : *ScaleSchema*; c : *Scale*; x, y, z : *Point-Grade*
  • ∃!sc : *ScaleSchema* • *scale*$(sc) = c$
  • (*inScale*$(x$,*scale*$(t))$ ∧ *inScale*$(y$,*scale*$(t))$ ∧ $\neg(x = y)$) ⇒ (*left-more*(x, y) ∨ *left-more*(y, x)))
  • *left-more*(x, y) ⇒ ∃!sc:*ScaleSchema* • *inScale*$(x$,*scale*$(sc))$ ∧ *inScale*$(y$,*scale*$(sc))$
  • *right-less*(x, y) ⇔ *left-more*(y, x)

  • ¬*left-more*(x, x) **%irreflexive**
  • (*left-more*(x, y) ∧ *left-more*(y, z)) ⇒ *left-more*(x, z) **%transitive**
  • (*left-more*(x, y) ⇒ ¬*left-more*(y, x) **%antisymmetric**
end

Let us now examine the degrees of integration and of topology preservation of this blend.

Integration. For measuring the integration of the blend, we identify a least general generalisation V_e of our original span V for which the colimit is epic. This amounts to making a minimal inner-space generalisation of HASSE and 3SCALE so that *each* entity of the colimit fuses entities projected from *both* input spaces. (Analogous to Section 4.2.1, this generalisation removes the structure that is written in black in Specification 9.)

The degree of integration of HASSE_3SCALE_BLEND is then

$$\mathrm{Int}(\text{HASSE\_3SCALE\_BLEND}) = \sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$$

where m : HASSE_$0^\dagger$ ↣ HASSE is the **Spec***-mono of the inclusion of HASSE_$0^\dagger$ into HASSE, and n and o are identity arrows; and where m' : HASSE_E ↣ HASSE and n' : 3SCALE_E ↣ 3SCALE are the **Spec***-monos of the inclusions of HASSE_E and 3SCALE_E into HASSE and 3SCALE, respectively, and o' is again the identity arrow (since the generic space is not generalised).

The similarity $\sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$ of span generalisations is calculated in terms of the generalisation steps of a given generalisation operator (see Definition 6). Assuming an operator that removes signature entities as in Section 4.2.1, and taking into account that HASSE_E is a generalisation of HASSE_$0^\dagger$, (i.e. there exists **Spec***-mono m'' : HASSE_E ↣ HASSE_$0^\dagger$ such that $m' = m \circ m''$), we have that:

$$\begin{tikzcd}
V_0 \arrow[dr, "id"'] & & V_e \arrow[dl, "\langle m'', n', o'\rangle"] \\
& V_0 \arrow[d, tail, "\langle m,n,o\rangle"'] & \\
& V &
\end{tikzcd}$$

And thus:

$$\begin{aligned}
&\sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle) \\
&\quad = \frac{1 + \lambda(\langle m, n, o\rangle)}{1 + \lambda(\langle m, n, o\rangle) + \lambda(id) + \lambda(\langle m'', n', o'\rangle)} \\
&\quad = \frac{1 + (\lambda(m) + \lambda(n) + \lambda(o))}{1 + (\lambda(m) + \lambda(n) + \lambda(o)) + \lambda(id) + (\lambda(m'') + \lambda(n') + \lambda(o))} \\
&\quad = \frac{1 + (6 + 0 + 0)}{1 + (6 + 0 + 0) + 0 + (41 + 11 + 0)} \\
&\quad = \frac{7}{59} \approx 0.1186
\end{aligned}$$

Topology. For measuring the topology preservation of the blend, we need to identify a minimal general generalisation V_m of our original span V for which the colimit is monic. Analogous to Section 4.2.1, this amounts to making a minimal outer-space generalisation of the cross-space correspondence between input spaces (by generalising the specification GENERIC of the span modelling the cross-space correspondence).

The degree of topology preservation of HASSE_3SCALE_BLEND is then

$$\mathsf{Top}(\text{HASSE\_3SCALE\_BLEND}) = \sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)$$

where m : HASSE_$0^{\dagger} \rightarrowtail$ HASSE is the **Spec***-mono of the inclusion of HASSE_$0^{\dagger}$ into HASSE, and n and o are identity arrows; and where now m' and n' are identity arrows (since input spaces are not generalised), while o' : GENERIC_M $\rightarrowtail$ GENERIC is the **Spec***-mono of the inclusion of GENERIC_M into GENERIC.

Assuming the same generalisation operator as above, and taking into account that the most general generalisation of V of which both V_1 and V_m are generalisations, is V itself, we have that:

$$\begin{tikzcd} V_0 \arrow[dr, tail, "\langle m,n,o\rangle"'] & & V_m \arrow[dl, tail, "\langle m',n',o'\rangle"] \\ & V \arrow[d, tail, "id"'] & \\ & V & \end{tikzcd}$$

And thus:

$$\begin{aligned}
&\sigma(\langle m, n, o\rangle, \langle m', n', o'\rangle)\\
&= \frac{1 + \lambda(id)}{1 + \lambda(id) + \lambda(\langle m, n, o\rangle) + \lambda(\langle m', n', o'\rangle)}\\
&= \frac{1 + 0}{1 + \lambda(id) + (\lambda(m) + \lambda(n) + \lambda(o)) + (\lambda(m') + \lambda(n') + \lambda(o'))}\\
&= \frac{1 + 0}{1 + 0 + (6 + 0 + 0) + (0 + 0 + 6)}\\
&= \frac{1}{13} \approx 0.0769
\end{aligned}$$

5. Discussion

We have initiated our formalisation of governing principles for analysing diagram sensemaking by addressing the Integration and Topology-Preservation principles. These principles hold a central position within conceptual blending. In Section 2.4, we have revisited these principles, aligning them with our understanding of Fauconnier and Turner's original propositions. This clarification aims to reduce ambiguities and facilitate their formalisation in Section 3.1.2, following an interpretation-independent approach based on the category-theoretical framework proposed by Schorlemmer and Plaza (2021). According to Fauconnier and Turner, "the impulse to achieve integrated blends is an overarching principle of human cognition" (Fauconnier & Turner, 2002, p. 328), emphasising the significance of the Integration principle.

Simultaneously, we strive to retain as much information as possible from the input spaces' structure, which is captured by the Topology-Preservation principle. As mentioned in Section 2.3, Fauconnier and Turner proposed additional governing principles for conceptual blending, including Compression, Pattern Completion, Promoting Vital Relations, Web, Unpacking, and Relevance. These principles are considered advantageous for reasoning and communication, offering cognitive benefits such as reduced computation or memory costs. However, Integration and Topology-Preservation exhibit a compelling duality when formalised within our category-theoretic framework, with integration evaluated through epic colimits and topology assessed through monic colimits. Consequently, we have prioritised the initial analysis of Section 4 on these two governing principles due to the aforementioned reasons.

5.1. Integration vs topology-preservation in diagram sensemaking

When examining the integration and topology-preservation measures computed for the various approaches to interpreting the ranked structure of the Hasse diagram shown in Fig. 4 and summarised in Table 1, we observe distinct values. The blend that exhibits the highest degree of integration between the diagram's geometry and the SCALE schema is the one that disregards the lines and solely focuses on the points, treating them as constituting multiple levels (referred to as the 'only-ranked' blend). This blend successfully integrates the ranked structure of the diagram with the SCALE schema. However, it also demonstrates the lowest degree of topology preservation since most of the information from the Hasse diagram is generalised away.

Table 1
Summary of the degrees of integration and topology preservation of the four different blends modelling alternative ways of making sense of the Hasse diagram of Fig. 4 as analysed in this section.

Blending	Integration	Topology preservation
Intended	0.0984	0.0833
Nongeneralised	0.0164	0.1429
Only-Ranked	0.7705	0.0185
Disjoint	0.0137	0.5385
Vertically-Levelled	0.1186	0.0769

Among the other blends we have examined, two of them stand out. Firstly, the 'intended' blend, where the Hasse diagram is understood as organised into four horizontal levels while preserving the information about point-line connections, exhibits a higher degree of integration between the diagram's geometry and the SCALE schema. Secondly, the 'vertically-levelled' blend, which captures our interpretation of the diagram as structured into three vertical levels, also demonstrates a notable degree of integration. In comparison, the remaining blends we analysed, namely the 'nongeneralised' blend and 'disjoint' blend, display lower degrees of integration of the diagram's geometry with the SCALE schema.

According to Fauconnier and Turner, "inputs often have opposed topologies; [p]rojecting these topologies into the blend could create a disintegrated space" (Fauconnier & Turner, 2002, p. 329). We aimed to demonstrate this disintegration through the case of the 'nongeneralised' blend. By refraining from generalising the diagram geometry, we achieve better topology preservation compared to the 'only-ranked' and 'intended' blending cases, as we retain more structure. However, this comes at the expense of integration. This aligns with Fauconnier and Turner's assertion that "it is a corollary of the Integration principle that in such cases, selections and adjustments must be made to avoid a disintegrated blend". Fauconnier and Turner (2002, p. 329). For instance, the 'intended' blending case maintains consistency and exhibits greater integration by generalising elements of the diagram geometry that lead to inconsistencies in the blend.

As Fauconnier and Turner suggest, "governing principles ... often conflict" (Fauconnier & Turner, 2002, p. 327), which is evident between the Integration and Topology-Preservation principles, as reflected in our formalisation. An extreme example of nonintegration is illustrated in the case of the 'disjoint' blend, which prioritises topology preservation but lacks integration.

It is important to note that the alternative approach of interpreting the Hasse diagram as vertically levelled exhibits higher integration but lower topology preservation compared to the intended blending case. However, comparing these measures when cross-space correspondences differ can be misleading. Factors such as the smaller signature and fewer axioms in the 3SCALE specification compared to the 4SCALE specification influence the computation of integration and topology-preservation measures. Additionally, the SCALE schema is not the sole image schema relevant to our embodied understanding of a Hasse diagram.

Bourou et al. (2021a, 2022) have shown that image schemas like LINK, PATH, and VERTICALITY also play significant roles in reasoning with Hasse diagrams. Considering the integration and topology preservation of the VERTICALITY schema with the geometry of the

Hasse diagram may assist in guiding our blending process towards the 'intended' sensemaking of the diagram, organised by horizontal levels that increase from bottom to top, rather than the alternative vertically-levelled interpretation or the one that disregards all lines. A detailed analysis of the role of the Integration and Topology-Preservation principles when blending the geometry of the Hasse diagram with VERTICALITY and other image schemas that are relevant when making sense of the entire diagram, however, is outside the scope of this article.

5.2. Salience and grouping in diagram sensemaking

The integration network between image schemas and the geometry of a diagram reflects how we mentally structure this geometry to make sense of and reason with it. This cognitive structuring involves emphasising certain parts of the geometric configuration while disregarding others or combining multiple shapes into a single entity. These phenomena are present to varying degrees in all the blends analysed in this article.

The emphasis on certain structures is evident in the blending of the SCALE schema with only the points (along with their relative position predicates) of a Hasse diagram's geometry, highlighting them as salient for reasoning about the entire geometry as an ordered/ranked poset (the 'only-ranked' blend). In this case, the lines and other predicates are generalised away from the Hasse diagram's geometric space and are not included in the blend, representing a cognitive instance where they are not attended to. A similar but lesser degree of emphasis occurs in the intended blending case, where most elements of the input spaces are retained and projected into the blend.

The grouping of multiple shapes into a single structure, resulting in a new entity, can be observed in the intended, 'only-ranked', the 'nongeneralised', and the 'vertically-levelled' blending cases. Here, several points of the diagram's geometry are mapped to a single grade, integrating them as point-grades in the blend. The Integration principle promotes this grouping, while Topology-Preservation is increased when cross-space correspondences are one-to-one, avoiding the grouping of elements from the input spaces. Conversely, the disjoint blending case, which exhibits the highest degree of topology preservation among the analysed cases, represents a blend where no grouping occurs.

6. Related work

In this section, we discuss previous research on the computational implementation of conceptual blending that has explicitly addressed the formalisation of governing principles, and we compare it with our own proposal.

6.1. Alloy and GRIOT

One of the early implementations of conceptual blending and its governing principles is Alloy (Goguen & Harrell, 2004). This algorithm is based on the algebraic semiotics approach defined by Goguen (1999). The input spaces of the Alloy algorithm are theories defined in the algebraic specification language BOBJ (Goguen et al., 2000) and implemented as graph structures. This allows implementing the possible outer-space relations as binary trees, which the blending algorithm conducts a search on to select a mapping. This work has many commonalities with our implementation, as our category-theoretical framework builds on it; in both works, blends are based on colimits, the approach of algebraic specification is followed, and an algebraic specification language is used.

The authors require governing principles to select blends, but they consider that the principles of Fauconnier and Turner (2002), although rich, require human meaning, and are thus not easy to implement in a computational system. An exception is Topology, which addresses structure rather than meaning, and so it is captured as the commutativity property for each of the triangles (left/right) in the pushout diagram

$$\begin{tikzcd} & B & \\ I_1 \arrow[ur] & & I_2 \arrow[ul] \\ & G \arrow[ul] \arrow[ur] \arrow[uu] & \end{tikzcd}$$

Fig. 5. Pushout diagram modelling two input spaces I_1 and I_2, a generic space G, and a blend B. The latter corresponds to the apex of the colimit (here pushout) for the span $I_1 \leftarrow G \rightarrow I_2$.

modelling the conceptual blend (see Fig. 5). This is the only principle used in Alloy (Goguen & Harrell, 2004).

Alloy was later integrated into GRIOT to generate narratives in poetry (Goguen & Harrell, 2010). In this work, the authors use their own set of principles, which are computationally effective but limited. In addition to diagram commutativity, the number of constants and axioms of the input spaces that are preserved in the blend is quantified, and so is the degree of type casting.[5] A weighted sum of the principles is taken, and thresholds can be set, above which blends are deemed acceptable. Measures considered to decrease optimality can be negative. The authors also claim Unpacking (as defined by Fauconnier and Turner (2002)) is satisfied for the blends resulting from their algorithm.

Finally, the authors point out that the advantages of certain blends, such as those corresponding to highly metaphorical poems, may not be captured by any of the principles of Fauconnier and Turner (2002), nor by those they proposed themselves. Such blends may violate Integration, Web, Unpacking, Topology and Relevance, while they may have a high occurrence of type casting, so it might be necessary to formalise some additional principles and incorporate them into systems for computational creativity. Other criteria that Goguen and Harrell (2004) propose to take advantage of are: the fact that incongruence is not necessarily problematic in metaphors, the existence of personification, oxymoron, and metonymic tightening (relations between elements of the same input should become as close as possible within the blend).

Firstly, we concur with Goguen and Harrell (2004, 2010) that the principles of Fauconnier and Turner (2002) require human understanding and are hard to formalise computationally if they are not reinterpreted to a certain extent. In this article, we have opted for revisiting and formalising a couple of these principles, instead of proposing another set of them. Second, we see that the features of the blend used to assess the governing principles in these two papers roughly correspond to what we understand as Topology Preservation. Finally, additional principles discussed, such as metonymy and oxymoron, we believe are too specific to certain blends, e.g., in art, and in our work we opted for a general formalisation that may subsume other, more specialised, principles.

6.2. Divago and BlendVille

The first thorough implementation of a conceptual blending system is, to the best of our knowledge, Divago (Pereira, 2007). Two input spaces for blending are selected from a knowledge base of various micro-theories represented in Prolog, each of them consisting of a concept map (implemented as a semantic network, where nodes are concepts and arcs are binary relations), logical rules specifying what constitutes an incongruence for this concept, and some frames associated with it. The cross-space mapping is defined using a structure alignment algorithm based on spreading activation, which finds a partial one-to-one mapping between elements of the input domains,

[5] Type casting refers to when a constant in the blend gets a type that is not the same, or a subtype of, the one it had in the input space.

based on the identity of relational structure. Many possible blends can be generated from this mapping, so the selection of the final blend is done with a genetic algorithm. A weighted sum of the values of several governing principles for each blend serves as its fitness function.

The governing principles are discussed in detail by Pereira and Cardoso (2003). The degree to which frames are accomplished in the blend plays a prominent role in the formalisation of the governing principles. The accomplishment of a frame refers to the concept map of the blend having the same structure as the frames associated with the input spaces to this blend.

In the formalisations of Pereira and Cardoso (2003) we see that several principles, as described by Fauconnier and Turner (2002), require information about the meaning, or the purposes, of a particular blend. Indicatively, this occurs in the principles of Integration; when discussing what comprises incongruence, Promoting Vital Relations; when defining intensity for a particular relation, and Relevance; because certain frames are assumed to be goal frames for a given blend. Moreover, certain frames are assumed to be associated with a particular concept. The authors try to adapt these definitions in a way that works for their input spaces, giving rise to a very powerful but specialised system.

Following the footsteps of Divago, BlendVille is a computational conceptual blending system that uses evolutionary algorithms more exploring the blend space (Gonçalves et al., 2017); but unlike Divago, it selects and assesses blends not by way of governing principles as in Fauconnier and Turner (2002), but by proposing to evaluate topology, entropy, frame related informative measures and general informative measures. They base their choice of simplifying the number of governing principles on the study by Martins et al. (2016) where the authors suggest that five governing principles – Integration, Topology, Unpacking, Relevance, and Vital Relations – are enough for generating good blends

We offer a more abstract but general-purpose, mathematical foundation for formalising governing principles, linking them to our uniform model of conceptual blending, and we also focus on two significant governing principles—Topology and Integration. Another difference we note is that, while we put frames aside completely, they have a prominent role in Divago and BlendVille. Nonetheless, frames could be incorporated in our model as additional input spaces which do not get generalised, precisely as we do with image schemas in our case studies in Section 4. Finally, Topology seems to be the most unambiguous principle, as it has a relatively straightforward, and similar, formalisation in our work and in all aforementioned publications blending systems.

6.3. Cobble

Within the context of the COINVENT project (Schorlemmer et al., 2014), another implementation of conceptual blending has been proposed, which shares with the examples presented here the CASL-based representation of input spaces and the amalgam-based approach to blending (Bou et al., 2018). The blending system has been called Cobble, and it draws from several enabling technologies (Confalonieri et al., 2018). The computational framework has been applied to computational creativity in music harmonisation, mathematics, and formal methods in computer science (Eppe et al., 2015, 2018).

In this and other implementations of conceptual blending, important elements of the input spaces for a given integration network are predetermined by assigning manual priority indices to axioms, predicates, or operations. These indices are then used as input for various metrics aimed at assessing the structural criteria of integration networks, and they effectively reduce the search space for the specific problem at hand. Unlike the approach we have presented in this article, Cobble defines and formalises governing principles relative to the particular representation formalism it uses. Here we have been reinterpreting governing principles in the uniform, category-theoretic model of conceptual blending, aiming at a generic, representation-independent characterisation of these principles. Consequently, these representation-independent formalisations of governing principles are orthogonal to (and thus compatible with) representation-specific assignments of priority indices to elements of input spaces. Both approaches can contribute to assessing the value of a blend, and their relative importance will depend on the particular domain of application.

7. Conclusions and future work

We have introduced a formal framework for understanding the sensemaking process of diagrams, treating them as conceptual blends of image schemas with the geometry of the diagrams. We represented these blends as amalgams and formalised them using category-theoretical concepts. Our main contribution was the incorporation of a category-theoretic characterisation of governing principles for conceptual blending, with a specific focus on Integration and Topology Preservation. This allowed us to quantitatively measure the degree to which alternative blends satisfy these governing principles. By using these principles in our framework, which are proposed to guide us towards cognitively useful blends, we aimed to define, formalise, and implement them in a way that captures the tradeoffs between them and makes them amenable to algorithmic systems.

We believe that our work holds potential for applications in diagrammatic reasoning. Assuming that the interpretation of a diagram by an observer can be modelled as a conceptual blend of image schemas with the diagram's geometry, we anticipate that a likely interpretation would be represented by a blend with image schemas that maximally satisfy the governing principles among other possible blends. In future research, we plan to explore potentially interesting conceptual blends from a pool of image schemas that have been blended with the geometry of a diagram. Such a task requires an efficient way to explore the search space of possible amalgams.

Our objective is to use the formalised governing principles to guide this search. With the aid of such an algorithm, it becomes possible to assess the efficacy of a diagram and gain a qualitative understanding of the sensemaking process associated with it. If a diagram is cognitively effective, there will exist image schemas that can be blended with its geometry, resulting in a blend that satisfies the governing principles and facilitates valid inferences about the diagram's semantics. This framework has already been applied to various diagrams, allowing us to model different syllogisms involving them Bourou et al. (2021a, 2021b, 2021c), as well as compare their cognitive effectiveness (Bourou et al., 2022).

All of these goals are facilitated by having formal descriptions of image schemas as primitives that capture our embodied experiences. Image schemas have been proposed to structure our perception and bridge it with abstract thought (Mandler, 2004; Mandler & Pagán Cánovas, 2014; Fauconnier & Turner, 2002, pp. 104–105). While there is no definitive list of image schemas, they are conceptually and computationally simple. This makes them suitable for modelling and implementation in computational systems, such as detecting affordances in robotics (Pomarlan & Bateman, 2020; Pomarlan et al., 2021; Shanahan et al., 2020; Thosar et al., 2021) or studying spatial meaning across languages (Gromann & Hedblom, 2017).

In this article, our aim was to present a formal exploration of the governing principles of conceptual blending and their potential applications. We do not intend to provide a definitive description of governing principles at this stage. Our objective is to concretise the details of conceptual blending theory, which exists on the conceptual level so that it can be effectively utilised by other researchers. We have approached this theory formally using category theory and attempted to reinterpret and incorporate the governing principles within our framework to provide an understanding of the general characteristics of blends that could be leveraged for selecting effective blends. We believe that image schema and conceptual blending theory hold significant value for the development of computational cognitive systems. However, their conceptual and occasionally ambiguous nature hinders progress, which is why we have endeavoured to address this issue in our work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix. Category theory

In this section, we recall the basic category-theoretic notions we use in our work, drawing from Mitchell (1965) and Pierce (1991) for the basic definitions, and from Jay (1991) and Robinson and Rosolini (1988) for categories of partial arrows, with some minor adjustments to terminology and notation.

A.1. Categories, objects, and arrows

A *category* $\mathscr{C}$ comprises:

- a collection of *objects*;
- a collection of *arrows* (often called morphisms);
- operations assigning to each arrow f an object $dom\, f$, its *domain*, and an object $cod\, f$, its *codomain* (we write $f : A \to B$ of $A \xrightarrow{f} B$ to show that $dom\, f = A$ and $cod\, f = B$; the collection of all arrows with domain A and codomain B is written $\mathscr{C}(A, B)$);
- a composition operator assigning to each pair of arrows f and g, with $cod\, f = dom\, g$, a *composite* arrow $g\circ f : dom\, f \to cod\, g$, satisfying the following associative law: for any arrows $f : A \to B$, $g : B \to C$, and $h : C \to D$, $h\circ(g\circ f) = (h\circ g)\circ f$;
- for each object A, and *identity* arrow $id_A : A \to A$ satisfying the following identity law: for any arrow $f : A \to B$, $id_B\circ f = f$ and $f\circ id_A = f$.

For the purposes of the work presented in this article, we will assume (as in Pierce (1991)) that collections of objects and arrows are sets, operations dom, cod, and $\circ$ are set-theoretic functions, and that equality is set-theoretic identity. These categories are called *small* categories.

The *opposite* of a category $\mathscr{C}$, denoted $\mathscr{C}^{op}$, has the same objects as $\mathscr{C}$ but all its arrows reversed.

A category $\mathscr{D}$ is a *subcategory* of $\mathscr{C}$ if each object of $\mathscr{D}$ is an object of $\mathscr{C}$; for all objects A and B of $\mathscr{D}$, $\mathscr{D}(A, B) \subseteq \mathscr{C}(A, B)$; and composite and identity arrows are the same in $\mathscr{D}$ as in $\mathscr{C}$.

An arrow $f : A \to B$ is an *isomorphism* if there is an arrow $f^{-1} : B \to A$, called the *inverse* of f, such that $f^{-1}\circ f = id_A$ and $f\circ f^{-1} = id_B$. The objects A and B are said to be *isomorphic* (and we write $A \cong B$) if there is an isomorphism between them.

An arrow $f : B \to C$ of a category $\mathscr{C}$ is a *monomorphism* (or "is a *mono*", or "is *monic*") if, for any pair of arrows $g : A \to B$ and $h : A \to B$ of $\mathscr{C}$, the equality $f\circ g = f\circ h$ implies that $g = h$. (In this case we will write $f : B \rightarrowtail C$.)

An arrow $f : A \to B$ of a category $\mathscr{C}$ is an *epimorphism* (or "is an *epi*", or "is *epic*") if, for any pair of arrows $g : B \to C$ and $h : B \to C$ of $\mathscr{C}$, the equality $g\circ f = h\circ f$ implies that $g = h$. (In this case we will write $f : B \twoheadrightarrow C$.)

Given a mono $f : A' \rightarrowtail A$, we shall call A' a *subobject* of A, and we shall refer to f as the *inclusion* of A' in A. In general, there is more than one mono from A' to A, so that whenever we speak of A' as a subobject of A we shall be referring to a specific inclusion mono f.

The *image* of an arrow $f : A \to B$ in category $\mathscr{C}$ is defined as the smallest subobject of B which f *factors through*; that is, the image is an inclusion mono $m : I \rightarrowtail B$ in $\mathscr{C}$ satisfying that there exists an arrow $e : A \to I$ such that $f = m\circ e$, and for every subobject $m' : I' \rightarrowtail B$ and arrow $e' : A \to I'$ such that $f = m'\circ e'$, there exists a unique arrow $v : I \to I'$ such that $m = m'\circ v$. (Necessarily the arrow v is monic.) When e is epic, we call the pair of arrows (e, m) an *epi-mono factorisation*.

The *union* of a family $\{A_i\}_{i\in I}$ of subobjects of A is defined as the subobject A' of A, denoted by $\bigcup_{i\in I} A_i$, which is preceded by each of the A_i (i.e., each A_i is also subobject of A'), and which has the following property: If, for an arrow $f : A \to B$, each A_i is carried into some subobject B' of B by f (i.e., there exists an arrow $f_i : A_i \to B'$ such that $f\circ m_i = n\circ f_i$, where m_i and n are the inclusion monos of A_i in A and B' in B, respectively), then A' is also carried into B' by f (i.e., there exists an arrow $f' : A' \to B'$ such that $f\circ m' = n\circ f'$, where m' is the inclusion mono of A' in A, denoted by $\bigcup_{i\in I} m_i$).

When necessary to disambiguate, we will explicitly mention with a prefix the category $\mathscr{C}$ a particular entity is part of, and thus talk of $\mathscr{C}$-objects, $\mathscr{C}$-arrows, $\mathscr{C}$-monos, etc.

A.2. Diagrams

A *diagram* in a category $\mathscr{C}$ is a collection of vertices and directed edges, consistently labelled with objects and arrows of $\mathscr{C}$, where "consistently" means that if an edge in the diagram is labelled with an arrow f and f has domain A and codomain B, then the endpoints of this edge must be labelled with A and B.

In this article we focus on diagrams with vertices and edges of particular shapes, namely V-diagrams, such as for instance:

$$\begin{tikzcd}
B & & C \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

and also W-diagrams, such as for instance:

$$\begin{tikzcd}
C & & D & & E \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] & & B \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

A.3. Universal constructions

A *product* of two objects A and B is an object $A \times B$ together with two projection arrows $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, such that for any object C and pair of arrows $f : C \to A$ and $G : C \to B$ there is exactly one mediating arrow $\langle f, g\rangle : C \to A \times B$ such that $\pi\circ\langle f, g\rangle = f$ and $\pi_2\circ\langle f, g\rangle = g$.

A *pullback* of the pair of arrows $f : A \to C$ and $g : B \to C$ is an object P (called *apex* of the pullback) and a pair of arrows $g' : P \to A$

and $f' : P \to B$ such that $f \circ g' = g \circ f'$; and if $i : X \to A$ and $j : X \to B$ are such that $f \circ i = g \circ j$, then there is a unique $k : X \to P$ such that $i = g' \circ k$ and $j = f' \circ k$.

Each universal construction has its dual, obtained by reversing the direction of arrows. In this paper, we also use the dual of the pullback:

A *pushout* of the pair of arrows $f : A \to B$ and $g : A \to C$ is an object P (called *apex* of the pushout) and a pair of arrows $g' : B \to P$ and $f' : C \to P$ such that $g' \circ f = f' \circ g$; and if $i : B \to X$ and $j : C \to X$ are such that $i \circ f = j \circ g$, then there is a unique $k : P \to X$ such that $i = k \circ g'$ and $j = k \circ f'$.

A pushout is a particular kind of the following universal construction (when the given arrows constitute a v-diagram):

A *colimit* for a diagram is an object C (called *apex* of the colimit) and a family of arrows $\{f_i : A_i \to C\}$ with an arrow from each of the objects A_i in the given diagram, such that, for every arrow $a : A_i \to A_j$ in the given diagram, $f_i = f_j \circ a$; and, if $\{g_i : A_i \to X\}$ is a family of arrows with an arrow from each of the objects A_i in the given diagram, such that, for every arrow $a : A_i \to A_j$ in the given diagram, $g_i = g_j \circ a$, then there is a unique $k : C \to X$ such that $g_i = k \circ f_i$.

A category $\mathscr{C}$ is said to ve *cocomplete* if there exists a colimit for every diagram in $\mathscr{C}$.

A.4. Categories of partial arrows

A pair of arrows $f : A \to B$ and $g : A \to C$ with the same domain constitute a *span* between B and C. When f is monic, we call it a *monospan* from B to C.

Monospans are a categorical abstraction of partial functions on sets, and when seen this way they are called *partial arrows*. The subobject as determined by the inclusion mono of the monospan is then called the *domain* of the partial arrow. When this inclusion mono is the identity, the arrow is a *total arrow*.

For partial arrows to constitute a category we need to have *inverse images* (i.e., pullbacks of monos) to be able to define composition of partial arrows. Monos satisfying this property are called *stable* monos. In the particular case of a pullback of two monos, the apex of the pullback is also referred to as the *intersection* of the subobjects represented by the monos, and given two subobjects A_1 and A_2 of an object A, we will write $A_1 \cap A_2$ for their intersection.

We write $\mathsf{Ptl}(\mathscr{C})$ for the category of partial arrows on $\mathscr{C}$. Often $\mathsf{Ptl}(\mathscr{C})$ is too big because the class of monos of $\mathscr{C}$ is too wide; thus, it is common to restrict the class of subobjects that are considered admissible as domains of partial arrows. We call such class a *realm* when its monos are stable and closed under composition and isomorphism; and we write $\mathscr{M}-\mathsf{Ptl}(\mathscr{C})$ for the category of partial arrows, when the monos of the monospans representing its arrows are taken only from the realm $\mathscr{M}$.

A.5. Functors and presheaves

Given two categories $\mathscr{C}$ and $\mathscr{D}$, a *functor* $F : \mathscr{C} \to \mathscr{D}$ is a map from $\mathscr{C}$-objects to $\mathscr{D}$-objects and from $\mathscr{C}$-arrows to $\mathscr{D}$-arrows, such that for all $\mathscr{C}$-objects A and composable $\mathscr{C}$-arrows d and g:

- $F(id_A) = id_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

The functors from a category $\mathscr{C}$ to a category $\mathscr{D}$ constitute themselves a category $\mathbf{Func}(\mathscr{C}, \mathscr{D})$ called *functor category*.

A *presheaf* on a category $\mathscr{C}$ is a functor $F : \mathscr{C}^{op} \to \mathbf{Set}$, where $\mathbf{Set}$ is the category of sets and functions. The category $\mathbf{Psh}(\mathscr{C})$ of *presheaves* on a category $\mathscr{C}$ is the functor category $\mathbf{Func}(\mathscr{C}^{op}, \mathbf{Set})$. Its colimits for a diagram (in the functor category) are constructed *pointwise*, i.e., by constructing the colimit at each value of the diagram for each object in $\mathscr{C}$.

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